Participating Mortgages and the Efficiency of Financial Intermediation

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Abstract

The aim of this paper is to establish a basic framework of financing with a highly flexible instrument, of Participating Mortgages (PMs), to improve the efficiency of the financial system. We distinguish these from convertible mortgages and derive closed-form solutions to price a whole framework of facilities under the PM umbrella. As most contracts are of definite tenure, our contribution includes finite maturity pricing formulae, which are scarce in real estate finance. We also focus on random tenure mortgages, which occur in the context of default and pre-payment risk. Finally, we conclude our study with a public policy implication of employing PMs as workout loans especially in the ongoing sub-prime crisis.

Keywords: Participating Mortgage (PM), Shared Appreciation Mortgage (SAM), Shared Income Mortgage (SIM), Shared Equity Mortgage (SEM), Profit Caps and Floors.

JEL: C63, D11, D14, D92, G13, G21, R31.

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1 Introduction

The efficiency of the financial intermediation system is of utmost importance. This is because financial intermediaries have the capacity to render the economy vulnerable to risk, as they connect real estate prices with the macro-economy (see Glaeser (2000)).\(^1\) The intense pressure on the global financial system stemming from the US sub-prime mortgage crisis highlights the inefficiency of the system. This crisis was triggered by the mis-selling of adjustable rate mortgages (ARMs) with low introductory (teaser) rates by unscrupulous lenders. This was done to qualify the poor (and minorities), who were perceived as being too risky and therefore shunned by prime lenders (see Ip and Paletta (2007); and Knight (2007)).\(^2\) These underprivileged American families, thus, received a rude shock in terms of high interest rates at the expiration of the introductory offer. This caused them to fall behind their mortgage obligations and become subject to foreclosure proceedings. Estimates are that around 2.4 million American families will lose their homes, thereby wiping out their meagre life savings (in the form of equity), thus exposing them to a life of “abject poverty” (see Economist (2007b); Gapper (2007); and Mason and Rosner (2007)).

The International Monetary Fund estimates the total losses and write-downs of the sub-prime woes to be in the vicinity of $1 trillion (see Guha (2008)). Its repercussion is felt across the globe and includes the following: (i) Loss in Market Value of around $273 billion of bonds associated with sub-prime mortgages devastating the capital bases of financial institutions on both sides of the Atlantic (see Economist (2008); and Guha (2008)); (ii) Failure of more than 40 sub-prime lenders (see Authers (2007); Cowan and Cowan (2004) study default correlations in a portfolio of sub-prime residential loans); (iii) Increase in supply of homes for sale (due to repossessions) thereby depressing their prices and negatively impacting on the construction sector and sales of durable goods (see Economist (2007a); Spector (2007)); (iv) Refusal of U.S. government sponsored agencies such as Fannie Mae and Freddie Mac to deal with existing subprime lenders for purchase of loans or for serving as a primary service provider (see Economist (2007c)); (v)

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\(^1\)Glaeser’s (2000) hypothesis is borne by the fact that: (a) Regional home bubbles have negative impact on residential investment and thus aggregate output (see Higgins and Osler (1998)); (b) A sharp fall in house prices leads to a reduction in consumption (through the wealth effect—see Case et al. (2005)); (c) A significant decline in real estate prices leads to foreclosures and losses for lenders, thus straining the banking system (see Case (2000)); (d) Endogenous developments in the credit markets are amplified and transmitted to the macro-economy (through the financial accelerator effect—see Bernanke et al. (1999); and Aoki et al. (2004)).

\(^2\)In general, ARMs are not appropriate for households with a large mortgage, volatile income, high default cost or low moving probability (see Campbell and Cocco (2003) and Campbell (2006)).
Scrutiny of the remaining subprime lenders by state and federal regulators (see Ip and Paletta (2007)); (vi) Tightening of credit to firms (in other industries, hedge funds, private equity groups etc.) leading to a recession in the U.S. and a decline in value of American assets (see White et al. (2007)); (vii) Increase in capital market volatility ensuing from the systemic problems in the U.S., spreading overseas, and crimping world growth (see Economist (2007a); Gapper (2007)). Stability of the financial system is a very important issue in many parts of the world, for example for Central and Eastern European countries, which are preparing their economies for adoption of the euro currency. These countries have recently experienced a lending boom to households unseen before, mostly in the form of mortgage loans (see Backé and Wójcik (2008)).

There are various attempts to revive the economy by improving the sharing rules in the depressed real estate sector. One possible solution explored in this paper is the employment of Participating Mortgages (PMs). These mortgages allow borrowers to remain owners of their property at the expense of sharing their upside payoffs with the financier.

Participating mortgages, also known as participation loans, are a family of facilities secured by real assets where the lender gets a proportion of payoffs either in the operating stage or liquidating stage or both. The borrower trades off the participation feature of the loan with the lender by either having the coupon rate set below the prevailing market interest rate and/or being granted a higher loan to value ratio. Depending on the contract, usually non-standardized and negotiated over the counter, the lender participates in either gross or net operating income, cash flows after senior debt service or proceeds from sale of the property.

Participation mortgages can help reconcile diverging interests of financiers and investors, particularly in the context of construction loans. This is because it may take several years for a new development to reach sufficient cash flow levels for repaying coupons of a traditional fixed-rate mortgage (or loan). Investors can obtain financing which would be impossible to obtain otherwise, given low initial level of cash flows. While being exposed to increased default risk during the construction phase, lenders are normally excluded from sharing the upside potential of a project (in a traditional fixed-rate mortgage). By offering participation clauses, lenders benefit from expected future payoffs in the operating and/or the liquidating phase.

Early studies focus predominantly on a special case of PMs, namely the Shared App-

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3Recently, both Glaeser (2007) and Summers (2008) encouraged using a particular type of PM, namely the shared-appreciation mortgage, to make “refinancing at lower interest rates more attractive for lenders,” thereby solving the subprime “lending mess.”
precipitation Mortgages (SAMs). On this front-line, contributions include French and Haney (1984) who used an option approach, improving on Dougherty, Villani and Van Order (1982) that didn’t have stochastic property prices. Page and Sanders (1986) study the impact on SAMs of house and interest rate risks. Murphy’s (1990) practical analysis of SAMs, while not extending the pricing methodology, provides some simulations. Recently, Sanders and Slawson (2005) used stochastic house prices coupled with stochastic interest rates to numerically simulate SAM prices.

Despite many valuable features stated above, the academic literature on the universal models for pricing general PMs is limited (see Alvayay et al. (2005)). Ebrahim (1996) establishes Pareto-superiority of participating mortgages in an overlapping generations framework. Ebrahim and Sikandar (2007) argue that participating and convertible securities are characteristic of the developed stage of a financial system. The participating features of mortgages mitigate agency costs thereby enhancing the value of the underlying property. This perspective is also supported by the practitioner literature, which highlights the benefit of affording property of higher value earlier in the life cycle, something which is accomplished by the employment of PMs (see Caplin et al. (2007)).

Different variations of participating mortgages are defined by Ebrahim and Sikandar (2007). They include the following forms observed in practice:

1. Shared Income Mortgage (SIM);
2. Shared Equity Mortgage (SEM);

These different contracts have been schematically represented on Figure 1 as a function of three factors and associated variables:

1. Income from Operations or Renting, represented by profit flow $P$;
2. Time Value, represented by interest rate $i$;
3. Capital Appreciation, represented by value of property $H$. 

PUT FIGURE 1 ABOUT HERE.

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In the case of Shared Income Mortgage (SIM) a positive income participation fraction $\theta_P > 0$ of profit flow $P$ is traded-off against interest rate $i$ required by the lender. Typically, this type of contract does not involve participation in capital appreciation. Consequently, the participation ratio related to appreciation of the property value is set to zero ($\theta_H = 0$).

In the case of Shared Equity Mortgage (SEM) the financier is a co-owner of the property. The lender shares risks and can be rewarded by a positive participation $\theta_P > 0$ of income from operations as well as by positive participation in capital appreciation $\theta_H > 0$. In this type of contract it is possible to completely exclude coupon payments, so that the required interest is set to zero ($i = 0$).

In the case of Shared Appreciation Mortgage (SAM), a positive participation fraction $\theta_H > 0$ of appreciation is traded-off against the required interest rate $i$. Typically, this type of contract does not involve participation in the rental income component. Consequently, the property income participation ratio is set to zero ($\theta_P = 0$).4

A PM is generally more appealing than a convertible mortgage since: (i) it is more versatile than a convertible facility as illustrated by its variants (given by SIM, SEM and SAM) described above; (ii) an investor retains control of the underlying project over the tenure of the PM unlike in the case of a convertible security where conversion to equity in good states of the economy dilutes control rights.5

The purpose of this paper is to establish a general framework of financing with a highly versatile instrument (such as a PM) to improve the efficiency of the financial system and link the variants of this instrument on a single platform. To this end, we price the variants in a closed-form solution (using profit caps) and discuss their employment as work-out loans in the ongoing sub-prime crisis.6

We implicitly assume the existence of an information architecture espoused in Levine et al. (2000). That is, an economy where property rights, foreclosure procedures (needed for real estate to serve as collateral) and accurate methods of valuing property are well

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4A SAM plays a vital role of increasing social welfare of home-owners, whose assets are highly concentrated in their residence with excessive amounts of debt. This is due to portfolio diversification which reduces the allocation of equity along with the periodic obligation of servicing the debt (see Dougherty et al. (1982)). See also Freiberg (1982), and Sanders and Slawson (2005) for more information.

5The payoffs of original equity holders are diluted in the good states of the economy for both PMs and convertible mortgages.

6Pricing a PM in a closed-form solution endows us with the following benefits: (i) Model calibration (esp. approximate and numerical models); (ii) More precise numerical approximations; (iii) Common ground for market participants to discuss levels of basic parameters such as implied volatility; (iv) Key factor in real time trading systems; and (v) Simulation to better understand practitioner and academic issues.
established. In this setting, we employ the closed-form solutions derived in Shackleton and Wojakowski (2007). We improve the efficiency of the financial system by pricing the variants of a PM (using profit caps) and discussing their employment as workout loans in the ongoing sub-prime crisis.

The paper is organized as follows. In the next section, we discuss the assumptions behind our participating mortgage model. The following three sections focus on Shared Income (SIM), Shared Equity (SEM) and Shared Appreciation (SAM) mortgages respectively. We then discuss several extensions and illustrative examples. The last section concludes.

2 The model

Assuming that agents are risk-neutral or, equivalently, that trading in underlying assets is somehow possible so that hedging is feasible, the profit process can be written as

\[ dP_t = (r - \delta) P_t dt + \sigma P_t dZ_t \] (1)

where \( Z_t \) is the corresponding Brownian motion under the risk-neutral measure \( Q \), \( r \) is the risk-free interest rate and \( \delta \) is a constant cash yield, which is analogous to the dividend rate in case of a stock. Depending on context: commercial or investment property, the cash flow \( P \) can be interpreted as income from operations or renting. A more intrepid interpretation includes salary income as source of cash flow \( P \) in the case of a household. The total current value of risky profit flow can be computed as a discounted risk-neutral expectation and is given by

\[ A_0 = \int_0^{\infty} e^{-rt} E\left[ P_t \right] dt = \frac{P_0}{\delta} \] (2)

Throughout this paper we adopt the convention that \( E\left[ x \right] \) is the expectation taken at time \( t = 0 \) under the equivalent martingale measure \( Q \) i.e. \( E\left[ x \right] = E_Q\left[ x \mid \mathcal{F}_0 \right] \) and \( E_t\left[ x \right] = E_Q\left[ x \mid \mathcal{F}_t \right] \).
$E^Q [x | \mathcal{F}_t]$, where $\mathcal{F}_t$ is information available at time $t$. In our context all mortgage contracts are of finite maturity, while the profit flow $P_t$ is assumed to be of infinite duration i.e. the support of $t$ is $[0, \infty)$. Note, however, that the total current value $A_t$ is driven by the same dynamics as $P_t$ i.e. (1). Furthermore, because $P_0 = \delta A_0$, the current profit cash flow $P_0$ can be represented as a constant proportion $\delta$ (dividend or cash flow yield) of it’s present value.

We will assume that a property, the value of which is $H_t$, is capable of generating cash flow $P_t$. The real estate value $H_t$ need not be perfectly correlated with the cash flow $P_t$ or equal to the present value thereof. The property value $H_t$ is assumed to be driven by a process which can be correlated to profit cash flow $P_t$

$$dH_t = (r - \delta_H) H_t dt + \sigma_H H_t dZ^H_t$$ (3)

where $\delta_H$ can be explained as the rental rate. Such interpretation of the cash yield parameter $\delta_H$ is typical in the context of housing finance, as e.g. in Capozza, Kazarian and Thomson (1998). A more general meaning of $\delta_H$ involves a service flow from using the house or warehouse over time, as in Kau, Keenan, Muller and Epperson (1992). The $\sigma_H$ is then the volatility of real estate, possibly different from the dispersion parameter $\sigma$ of the profit cash flow process $P_t$. In (3), $Z^H_t$ represents the risk-neutral standard Brownian motion process driving real estate prices. For simplicity, we will assume that the correlation of real estate price returns with the stochastic component $dZ_t$ driving profit cash flows is equal to $\rho = 0$ so that the instantaneous risk-neutral expectation of $dZ_t dZ^H_t = \rho dt$ is also equal to zero.

In the case of commercial mortgages we will assume that the financing of tools necessary to generate the profit flow is exogenously given, concentrating on financing the real estate property instead. It is, however, straightforward to extend our analysis to the case where the lender finances everything, buildings but also business equipment, for example. Alavyay et al. (2005) analyse such a situation. Their discrete-time formulation is constrained to the case of Shared Income Mortgages (SIM) only and involves finite sums (which requires numerical simulations) in place of our closed-form formulae.

Consider a conventional fixed-rate mortgage, known as a standard mortgage loan. At time $t = 0$ the investor makes an initial deposit $D_0$ against property valued at $H_0$. The

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*We assume that there are special techniques to decipher the market value of property from the appraisal based data (see Geltner et al. (2003)). In particular this implies that the market price of the property and its book value (or “estimate”) do not differ.
lender finances the net loan amount $Q_0$ and the time to maturity or tenure is $T$. The lender may wish to impose a maximum permitted loan-to-value ratio $L$ (typically $L < 1$, e.g. 90%) i.e. a minimum equity condition, such that

$$Q_0 = H_0 - D_0 \leq \bar{L}H_0$$  \hspace{1cm} (4)$$

implying that the initial deposit, $D_0$, must be greater than $D$, where

$$D = H_0 (1 - \bar{L}) \leq D_0$$  \hspace{1cm} (5)$$

The coupon schedule $\{a_t : t \in [0, T]\}$ is typically the other agreed clause in the loan. The value of the loan at time $t = 0$ must be equal to the discounted expected value of future loan cash-flows i.e. coupons and terminal balance

$$Q_0 = \int_0^T e^{-rt} E[a_t] \, dt + e^{-rT} E[Q_T]$$  \hspace{1cm} (6)$$

This is a very general expression. It must be satisfied by intermediate cash flows and by the terminal cash flow. The borrower is required to pay continuous interest coupon flow $a_t$ and reimburse the remaining principal $Q_T$ at maturity. Note that the infinitesimal coupon amount paid during time $dt$ is $a_t \, dt$, while the coupon flow rate (per unit of time) is $a_t$.

To begin with, consider non amortizing loans. The outstanding balance remains constant, equal to the initial loan amount $Q_0$, so that $Q_t = Q_0$ for all $t$, implying $Q_T = Q_0$. In this case the agreed coupon structure is typically constant in time and can be fully characterized by a single parameter $i$. It is known as cost of funds and is equal to the lender’s required, continuous coupon rate, such that

$$a_t = iQ_t = iQ_0 = a_0$$  \hspace{1cm} (7)$$

If the loan is riskless—in particular there is no default or prepayment risk—cash flows bear no risk and, as expected, simplifying expression (6) gives $i = r$, so that $a_t = rQ_0$.

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$^{10}$An algorithm to evaluate the optimal loan to value ratio ($\bar{L}$) and tenure of the facility ($T$) is available from the authors upon request.
3 Shared Income Mortgage (SIM)

In this case of participating mortgage the required coupon rate $i$ is reduced so that $i < r$ to accommodate lender’s participation. Instead, the lender is compensated proportionally to the excess intermediate profit flow $(P_t - K)^+$ where $K$ is the fixed profit threshold above which participation is payable and $x^+$ is the positive part function, equal to $x$ if $x > 0$ and to zero otherwise. Coupons $a_t$ are random and contingent on the level of profit flow

$$a_t = iQ_t + \theta_P (P_t - K)^+$$

where $\theta_P$ is the excess profit flow participation rate. Condition (6) then gives

$$Q_0 = \int_0^T e^{-rt} E[iQ_t] \, dt + \theta_P \int_0^T e^{-rt} E[(P_t - K)^+] \, dt + e^{-rT} E[Q_T]$$

To be more precise, this type of mortgage is often termed Shared Income Mortgage (SIM), because the financier subsidizes the interest component in return for a share in the income from operations.

If the loan is non-amortizing $Q_0 = Q_t = Q_T$ and (9) gives the following condition

$$Q_0 = \frac{iQ_0}{r} \left(1 - e^{-rT}\right) + \theta_P C(P_0, K, T) + e^{-rT} Q_0$$

where $C(\cdot, \cdot, \cdot)$ represents the profit cap function, written on the profit process $P_t$, with threshold $K$ over maturity $T$. The value of the profit cap can be computed using explicit closed form formulae established in Shackleton and Wojakowski (2007)

$$C(P_0, K, T) = \frac{P_0}{\delta} \left(1_{P_0 \geq K} - e^{-\delta T} \Phi \left(\frac{d^K_{0, P_0}}{\delta}\right)\right) - \frac{K}{r} \left(1_{P_0 \geq K} - e^{-rT} \Phi \left(\frac{d^K_{0, P_0}}{r}\right)\right) + A(b, a) P_0^b \left(1_{P_0 \geq K} - \Phi \left(\frac{d^K_{b, P_0}}{b}\right)\right) - A(a, b) P_0^a \left(1_{P_0 \geq K} - \Phi \left(\frac{d^K_{a, P_0}}{a}\right)\right)$$

where $\Phi$ is the cumulative normal probability distribution function and $1_x$ is the indicator function equal to 1 if condition $x$ is true and equal to zero otherwise. The four coefficients
\( d_{\beta}^{K,P_0} \) for different \( \beta \)'s can be computed using

\[
d_{\beta}^{K,P_0} = \ln P_0 - \ln K + \left( r - \delta + \left( \beta - \frac{1}{2} \right) \sigma^2 \right) T
\]

where parameters \( \beta \in \{ b, 0, 1, a \} \) represent the elasticity of each of the four components in the formula (11). Parameters \( a, b \) and the constant of integration \( A \) are given by

\[
a, b = \frac{1}{2} - \frac{r - \delta}{\sigma^2} \pm \sqrt{\left( \frac{r - \delta}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}}
\]

\[
A(a, b) = \frac{K^{1-a}}{|a - b|} \left( \frac{b}{r} - \frac{b - 1}{\delta} \right)
\]

Since formulae for all components are available in closed form, much progress is possible in determining dependence of relevant contract variables on input parameters. Having set the problem up, these comparative statics remain one of the main objectives of the paper.

When participation is limited to intermediate profit cash flows, condition (10) provides possible combinations \( \{ i, \theta_P \} \) between reduced interest and participation rate. Furthermore, the lender may fine-tune the loan by offering higher or lower maximum loan-to-value ratios \( \bar{L} \). Assuming that the constraint (4) is fully binding, so that \( Q_0 = \bar{L}H_0 \), where \( H_0 \) is the current market value of the property, gives

\[
i = r \left[ 1 - \frac{\theta_P C(P_0, K, T)}{\bar{L}H_0 (1 - e^{-rT})} \right]
\]

The interpretation of this result is straightforward. The contractual interest rate \( i \) on the mortgage can be traded-off against a positive participation rate \( \theta_P > 0 \). The higher the participation, the greater the reduction in interest rate.\footnote{By contrast, variable tenor mortgages described in Chow et al. (2000) have the capacity to only cap the contractual interest rate by automatically stretching the maturity of the contract. Moreover, as the authors point out, industry practice is to introduce a cap on the tenor, which has the effect of rendering such contracts similar to variable rate mortgages and therefore losing most of their attractiveness.}

\footnote{In case of SIM, a very high \( \theta_P \) can introduce the risk of moral hazard, which is opposite to the one elaborated in Shiller and Weiss (2000). Here, an investor can undertake extensive capital improvement to under-represent income and convert the excess income to capital appreciation (which is not shared with the financier).} This trade-off, if used optimally, has the capacity to reduce defaults and help improve the overall efficiency of the financial
system.

The reduction in interest rate can be moderated by higher loan-to-value ratio $L$. For sufficiently low $L$ and sufficiently high participation $\theta_P$ it is possible to achieve $i = 0$. At this point all cash flows to the financier become uncertain, contingent on profit flow and there are no fixed interest payments. Fixed income flow is substituted for a risky stream. When $i$ becomes negative $i < 0$, the lender pays the entrepreneur to extract a portion $\theta_P$ of random excess profit flow $(P_t - K)^+$.

It is relatively easy to evaluate the impact of increased time to maturity on the contractual interest rate $i$ by computing the partial derivative

$$\frac{\partial i}{\partial T} = \frac{r\theta_P}{LH_0} \left[ \frac{\partial C}{\partial T} - \frac{re^{-rT}}{1-e^{-rT}} C \right]$$

(16)

where $C = C(P_0, K, T)$ is the profit cap given in closed form by (11). The term in brackets contains the theta hedge ratio of the cap. It is equal to the expected, discounted terminal caplet payoff on the (unknown) flow $P_T$ at time $T$ struck at $K$ (i.e. $(P_T - K)^+$ discounted at rate $r$), which corresponds to evaluating at time $T$ the integrand in the second integral appearing in (9). Evaluating the risk-neutral expectation of the terminal excess profit flow produces the Black-Scholes (1973) formula for European call and is positive

$$\frac{\partial C}{\partial T} = e^{-rT} E\left[ (P_T - K)^+ \right] = \text{c} (P_0, K, T)$$

(17)

$$= P_0 e^{-\delta T} \Phi \left( d_1^{K,P_0} \right) - Ke^{-rT} \Phi \left( d_0^{K,P_0} \right) > 0$$

(18)

This is because the cap in equation (11) is a time integral of Black-Scholes options (18). The theta is positive in the sense that as the horizon $T$ gets longer by $\Delta t$, the profit cap $C$ becomes more valuable by $c (P_0, K, T) \Delta t$. The impact of maturity $T$ on $i$ can thus be obtained by establishing the sign of the term in brackets in (16).

For infinite maturity loans $C(P_0, K, \infty)$ does not depend on $T$, implying

$$\lim_{T \to \infty} \frac{\partial C}{\partial T} = 0 \quad \text{and therefore} \quad \lim_{T \to \infty} \frac{\partial i}{\partial T} = 0$$

(19)

as expected i.e. for $T \to \infty$, the contractual rate $i$ will gradually converge to some $i_\infty$ given

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13A caplet has a payoff which is mathematically identical to the payoff of a call option. The difference is that the caplet has for underlying a variable which is a continuous flow ($P_t$ in our case) in place of a stock for the call option.
\[ i_\infty = r \left[ 1 - \frac{\theta P}{TH_0} C(P_0, K, \infty) \right] \]  

(20)

In the example represented on Figure 2 the continuum of caplets on profit flow is initially deeply out-of-the-money \((P_0 \ll K)\). However, it is expected that participation will kick in in the future. Under assumptions of exponentially growing cash flow at the rate \(r\) (10% in our example) it will reach the threshold \(K = 1,000,000\) at \(t_K \approx 13.9\) years. Again under the same assumptions, the excess cash flow should be comparable to the property price at about \(t_H \approx 14.8\) years. It is clear from Figure 2 that, independently of the rate of participation \(\theta P\), the lender is able to grant a decent discount only if the maturity \(T\) of the loan exceeds both \(t_K\) and \(t_H\). Once \(P_t \geq K\), at this future point in time it will be further expected that the level of available cash flow to share will continue to grow exponentially. The higher the participation level \(\theta P\) and the longer the maturity \(T\), the higher the discount below \(r = 10\%\) the lender is willing to offer. If the maturity \(T\) of the loan is too short, the excess profit cash flow is zero, the discount is almost inexistent and so \(i \approx r\).

Alternatively, the participation rate \(\theta P\) can be expressed in terms of the contractual interest rate \(i\)

\[ \theta P = \frac{TH_0 (1 - e^{-rT})}{C(P_0, K, T)} \left(1 - \frac{i}{r}\right) \]  

(21)

Clearly, higher loan-to-value ratios \(L\) necessarily imply higher participation rates \(\theta P\). Similarly, the lower the contractual interest rate \(i < r\), the higher the participation \(\theta P\). Furthermore, impact of variations in time to maturity \(T\) or threshold profit flow level \(K\) can be assessed by employing appropriate sensitivities of the cap \(C\) with respect to underlying variables, which are readily available in closed form (see Shackleton and Wojakowski (2007) for hedge ratios).

Figure 3 illustrates the dependence of the participation ratio \(\theta P\) on maturity \(T\). It confirms intuition that, the longer the maturity of the loan, the lower the participation required to grant the same discount. Similarly, the higher the discount granted the higher

14See note 13
the required participation. The dashed line represents the highest discount (no interest charged \( i = 0 \)) and thus participation here must dominate other offers, for the time to maturity of mortgage loans kept constant. For loans of maturity \( T = 40 \) years it is sufficient to participate at the level \( \theta_P \approx 51\% \) in order to generate full discount \( (i = 0) \). On the other hand a logarithmic scale is used for \( \theta_P \) in order to illustrate the fact that for shorter maturity loans even participation parameter substantially higher than one \( \theta_P \gg 1 \) may prove insufficient to compensate for any given discount \( r - i \).

4 Shared Equity Mortgage (SEM)

Here, the financier is the co-owner of the property. We assume that the borrower needs \( H_0 \) at the onset in order to start the commercial activity or make a step on the property ladder. The lender pays the deposit \( D_0 \) (in fact this is not *stricto sensu* a deposit, but is his initial share) and lends \( Q_0 \) to the borrower. Note in particular that \( Q_0 < H_0 \). If coupon payments are waived \( (i = 0) \), the financier will require different forms of compensation. These are provided by participation in the profit flow \( P \) above some threshold level \( K \) as well as participation in the appreciation of the property value \( H \). Intermediate cash flow rate is \( \theta_P (P_t - K)^+ \). The terminal cash flow consists of the remaining capital \( Q_T \) plus the fraction \( \theta_H \) of the appreciation, if positive, in the value of property \( (H_T - H_0)^+ \), payable to the lender. Condition (6) then gives

\[
Q_0 = \theta_P \int_0^T e^{-rt} E \left[ (P_t - K)^+ \right] dt + e^{-rT} E [Q_T] + \theta_H e^{-rT} E \left[ (H_T - H_0)^+ \right] \tag{22}
\]

Two option components are present here. The first is linked to generation of intermediate profits by the occupier and is a continuous sum of discounted intermediate payments. The second, discrete and terminal optional component, uses real estate appreciation as ultimate mean of mortgage repayment.

For non-amortizing mortgages (22) gives

\[
Q_0 = \theta_P C (P_0, K, T) + e^{-rT} Q_0 + \theta_H c (H_0, H_0, T) \tag{23}
\]

where \( C (\cdot, \cdot, \cdot) \) represents the profit cap function \([11]\), while \( c (H_0, H_0, T) \) represents a Eu-
European call option on real estate $H$ issued at the money. It can be computed as

$$c(H_0, H_0, T) = H_0 \left[ e^{-\delta_H T} \Phi \left( d_{1,0,H_0}^{H_0,H_0} \right) - e^{-rT} \Phi \left( d_{0,0,H_0}^{H_0,H_0} \right) \right]$$

(24)

where

$$d_{\beta}^{H_0,H_0} = \frac{r - \delta_H + \left( \beta - \frac{1}{2} \right) \sigma_H^2 T}{\sigma_H \sqrt{T}} \quad \beta = 0, 1$$

(25)

Since all quantities in (23) are easy to compute, analysis of a SEM is simple. In particular (23) allows expressing the profit participation ratio $\theta_P$ as function of real estate participation ratio $\theta_H$

$$\theta_p = \frac{L H_0 (1 - e^{-rT}) - \theta_H c(H_0, H_0, T)}{C(P_0, K, T)}$$

(26)

where it is assumed as usual that the debt capacity is set at $Q_0 = LH_0$.

Since all quantities appearing in (26) are positive, we can immediately see that the higher the real estate participation $\theta_H$, the lower the profit participation $\theta_p$. For a given real estate participation $\theta_H$ ratio, sensitivities of profit participation $\theta_P$ to tenure $T$, interest rate level $r$, loan-to-value ratio $L$ or house price level $H_0$ can be computed in closed form. For example the sensitivity of the profit participation $\theta_P$ to house price level $H_0$ is given by

$$\frac{\partial \theta_P}{\partial H_0} = \frac{L (1 - e^{-rT}) - \theta_H \Delta_c(T)}{C(P_0, K, T)}$$

(27)

where $\Delta_c(T)$ is the delta of an at-the-money call option

$$\Delta_c(T) = \Phi \left( \left( r - \delta_H + \frac{1}{2} \sigma_H^2 \right) \frac{\sqrt{T}}{\sigma_H} \right)$$

(28)

Note in particular that $\Delta_c(T)$ is independent of the current level of e.g. a house price index $H_0$, because the mortgage is set initially at-the-money. These quantities may be very useful in structuring characteristics of a SEM contracts in practice. For example, in

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15Here too, one needs to be careful of the risk of moral hazard, where investor frivolously spends on capital improvements, which may not help in the appreciation of the property but may be for personal consumption. This situation is opposite to that reported in Shiller and Weiss (2000) but is akin to the case of “hubris” identified in the corporate takeover literature (see Roll (1986)).
order to immunize \( \theta_p \) to initial real estate price risk, \( \{ \theta_H, \theta_p \} \) should be set so that

\[
\begin{align*}
\theta_H^* &= \frac{\varpi (1-e^{-rT})}{\Delta_r(T)} \\
\theta_p^* &= \left[ H_0 - \frac{c(H_0,H_0,T)}{\Delta_r(T)} \right] \frac{\varpi (1-e^{-rT})}{C(P_0,K,T)} 
\end{align*}
\]

(29)

This result has been obtained by differentiating \( \theta_p \) given by equation (26) with respect to \( H_0 \) (see (27)), setting this derivative to zero and solving for \( \theta_H^* \). The corresponding \( \theta_p^* \) was obtained by inserting \( \theta_H^* \) in (26). We observe that \( \theta_H^* \) is then independent of \( H_0 \), while \( \theta_p^* \) is decreasing in \( H_0 \) (see Figure 4).

5 Shared Appreciation Mortgage (SAM)

In this case the lender subsidizes the interest component in return for a share in the appreciation when the property is sold. Under the assumption that the property can only be sold at maturity of the loan \( T \) we have

\[
Q_0 = \int_0^T e^{-rt} E \left[ i Q_t \right] dt + e^{-rT} E \left[ Q_T \right] + \theta_H E \left[ e^{-rT} (H_T - H_0)^+ \right] 
\]

(30)

where \( \theta_H \) is the share of real estate appreciation parameter and \( i \) is the below-market loan rate. Component \((H_T - H_0)^+\), structurally similar to a payoff of a call option issued at the money, represents the appreciation of the real estate (increase in value of e.g. a warehouse).

Since all quantities can be expressed in closed form, sensitivity analysis of a SAM is also rather simple. In particular, there is no sharing of intermediate option-like components. For non amortizing loans, (30) becomes

\[
Q_0 = Q_0^i \left( 1 - e^{-rT} \right) + Q_0 e^{-rT} + \theta_H c (H_0, H_0, T) 
\]

(31)

where \( c (H_0, H_0, T) \) represents an European call option on real estate \( H \) issued at the money and is given by (24) in closed form. It is straightforward to solve (31) for \( i \) or \( \theta_H \) and proceed with sensitivity analysis. When constraint (4) is fully satisfied \((Q_0 = L H_0)\), we obtain

\[
i = r \left[ 1 - \frac{\theta_H c (H_0, H_0, T)}{L H_0 (1 - e^{-rT})} \right] \]

(32)
where $H_0$ is the current market value of the property. This formula is analogous to (15). That is, the financier trades off interest rate with participation in the appreciation of the property. Here too, this trade-off has the capacity to reduce defaults and help improve the overall efficiency of the financial system.

The above closed-form solution improves on the existing literature which prices SAMs using numerical simulations (see Sanders and Slawson (2005)). It also helps us conceptualize the trade-off between the interest rate and appreciation parameters consistent with the statement of the above researchers as follows:

"Having a pricing model that simulates the tradeoff options within a base case SAM, which is also flexible enough to accommodate conditions or concerns, should aid in the understanding of basic SAMs and SAMs with slight contractual variations"

(Sanders and Slawson, (2005), pgs. 192-3).

Furthermore, the above property of a SAM is highly appealing for prospective low income home-owners. A SAM can help them qualify for a home loan better than an adjustable rate mortgage with a teaser rate. A SAM can also be employed as a workout loan in the on-going sub-prime mortgage crisis. Currently, US Treasury Secretary Paulson is encouraging the banking industry to come up with a pragmatic solution to avert foreclosure of homes, whose mortgage payments are 90 days or more past due (see Ishmael and Politi (2008)). A SAM is a viable alternative in areas where home prices are stable. It can also be pursued where home prices have declined and where the borrower is willing to decrease the exercise price of the appreciation option. This will certainly stabilize home prices and consequently help the global economy recover from the crisis.16

Finally17

$$\theta_H = \frac{LH_0 (1 - e^{-rT})}{c(H_0, H_0, T)} \left(1 - \frac{i}{r}\right).$$

(33)

The main difference is that (32) and (33) do not require using cap formula (11). Their analysis is thus simpler but similar conclusions will apply.

In the next sections we will discuss several possible extensions and illustrative examples of participating mortgages.

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16When pricing SAMs for a work-out loan, one should be careful using appraisal based information in the current pricing of properties. This is because the information from appraisals is erroneous and contradicts our initial assumption of accurate method of valuing property. It, thus, biases our pricing mechanism (see Sanders and Slawson, (2005)). One way around this issue is to avail of special techniques developed on extracting information on accurate valuation from appraisal based data (see Geltner et al. (2003)).

17Care must be taken to monitor adequate maintenance of property to avoid risk of moral hazard (see Shiller and Weiss (2000) as well as Sanders and Slawson (2005)).
6 General participating mortgage

So far we relied on the following particular features to define the specific forms of participating mortgages observed in practice:

1. No equity participation $\theta_H = 0$ for Shared Income Mortgage (SIM);
2. No fixed interest payments $i = 0$ for Shared Equity Mortgage (SEM);
3. No income participation $\theta_P = 0$ for Shared Appreciation Mortgage (SAM).

Relaxing either of these constraints leads naturally to the definition of General Participating Mortgage (PM). The generalized budget constraint becomes

$$Q_0 = \int_0^T e^{-rt} E [i_t Q_t] dt + \int_0^T \theta_{p,t} e^{-rt} E \left[ (P_t - K)^+ \right] dt + e^{-rT} E [Q_T] + \theta_H e^{-rT} E \left[ (H_T - H_0)^+ \right].$$

Note that unlike real estate participation $\theta_H$, interest payments $i_t$ as well as income flow participation $\theta_{p,t}$ may be given a time structure and need not be constant. In practice, this is often the result of a negotiating process between borrower and lender at the onset of the contract. If, on the contrary, interest and participation are constant in time, i.e. $i_t = i$ and $\theta_{p,t} = \theta_p$ and if the contract is non-amortizing, then the budget constraint (34) gives

$$\frac{LH_0}{r} \left( 1 - e^{-rT} \right) = \frac{LH_0}{r} \left( 1 - e^{-rT} \right) + \theta_P C (P_0, K, T) + \theta_H C (H_0, H_0, T)$$

where $C$ is the profit cap and $c$ is the call option. Both functions can be computed explicitly using (11) and (24) respectively. Condition (35) basically tells us that the present value of the net interest on the loan (left hand side) is attributable to the fraction $i$ of annuity (first term), participation $\theta_P$ in the income from operations an participation $\theta_H$ in the appreciation of property value. Again, it is easy to express $i$ as function of other parameters of the model to see that for positive participation ratios $\theta_P > 0$ and $\theta_H > 0$ the interest $i$ must be reduced below $r$

$$i = r \left[ 1 - \frac{\theta_P C (P_0, K, T) + \theta_H C (H_0, H_0, T)}{LH_0 \left( 1 - e^{-rT} \right)} \right] < r.$$
gage as one can opt for different combinations of $\theta_P$ and $\theta_H$ unlike the case of convertible where one is constrained to an equal value of them at the time of conversion. Here too, the above trade-off has the capacity to reduce defaults and help improve the efficiency of the financial system.\footnote{18}

In general, participating loans are not only applicable to the construction or infrastructure industry, where a gestation period allowing the newly developed structure to gradually build up its tenancy and subsequently its cash flows is necessary. It can also be used to finance a whole range of ventures such as technology start-ups. Finally, participating loans can also be used in intense negotiations between borrower and lender to structure a work-out loan.

7 Random tenure mortgages

It is more realistic to assume that $T$ is uncertain because a mortgage can be e.g. prepaid early. We will illustrate this problem for a shared appreciation, zero interest mortgage, in which case $\theta_H > 0$, $\theta_P = 0$ and $i = 0$. If the rate of growth of $H$ is expected to be sufficiently high,\footnote{19} a shared appreciation mortgage could offer no-interest loans ($i = 0$). When the property is sold at some random time $\tau$, gains accrued from the starting date of the loan are shared with the lender. The following condition must then hold

$$Q_0 = Q_0 \int_0^T e^{-rT} f(\tau) d\tau + \theta_H \int_0^T e^{-rT} E \left[ (H_\tau - H_0)^+ \right] f(\tau) d\tau$$

(37)

where $f(\tau)$ is the risk-neutral density of probability for a loan originated at time $t = 0$ being terminated at time $\tau > 0$. This approach is consistent with intensity-based modelling of default and prepayment risks.\footnote{20} Note that $e^{-rT} E \left[ (H_\tau - H_0)^+ \right]$ is given by a Black-Scholes formula evaluated at $\tau$. Structurally, the second integral in (37) is a weighted value

\footnote{18}{Here again, care must be taken to monitor adequate maintenance of property to avoid risk of moral hazard (see Shiller and Weiss (2000); and Sanders and Slawson (2005)).}

\footnote{19}{This would correspond to low or negative rental rate $\delta_H$ e.g. when homeowners invest cash to improve the property or mitigate depreciation.}

\footnote{20}{See, for example, the seminal work of Dunn and McConnell (1981) as well as more recent studies, e.g. Collin-Dufresne and Harding (1999), Gorovoy and Linetsky (2007) or Pliska (2005). By contrast, standard approaches typically involve foreclosures triggered by a specific house price and interest rate configuration (see e.g. Sharp et al. (2008, b) who use perturbation methods to numerically solve associated PDE systems). Occupation-time-triggered prepayment has also been suggested recently (see Sharp et al. (2008, a)).}
The participation ratio of a zero-interest shared appreciation mortgage is

$$\theta_{i=0}^{H} = \frac{1 - \int_{0}^{T} e^{-r\tau} f(\tau) d\tau}{C_f(H_0, H_0, T)} Q_0$$  \hspace{1cm} (39)$$

The above formula is an extreme version of a SAM which can be employed in lieu of a fixed or an adjustable rate mortgage, with or without a teaser rate. With no interest payments, this facility has the potential to cure the grim economic situation created by the defaulting sub-prime mortgages and thus improve on the efficiency of the financial system. We can also employ this facility in negotiations for a work-out loan as explained earlier.

8 Concluding remarks

An efficient mortgage finance system is an essential ingredient for economic development and subsequently its growth. Jaffee and Renaud (1997) articulate its importance as follows:

*Mortgage market development is likely to be a key factor in overall financial market development. In particular, an efficient mortgage market will act as a positive externality for the other capital markets, creating pressure for higher efficiency in these markets. On the other hand, a poorly functioning mortgage market is likely to ‘pollute’ other financial markets with its inefficiency.*

The ongoing sub-prime crisis and its debilitating effect on global capital markets are in accordance with their hypothesis.

This paper shed light on improving the efficiency of the mortgage finance system by describing an innovative facility in the form of a participating mortgage. This instrument has the capacity to manage risk at the micro-economic level with the potential for reducing default at the macro-economic level in accordance with the prognosis of Sheng (1997) and Renaud (2005). It is more appealing than a convertible mortgage as it is more versatile and retains its feature as debt through the tenure of the facility (without diluting control rights of its investors).
We proceed systematically by investigating different variants of *participating mortgages* (such as Shared Income, Shared Appreciation and Shared Equity Mortgages) to demonstrate that the closed-form *profit cap* formula allows tractability.

We thus contribute by evaluating finite maturity pricing formulae, which are scarce in real estate finance. We also focus on random tenure mortgages, which arise in the context of default and prepayment risk. Furthermore, we illustrate the general applicability of the variants of a PM in a range of projects ranging from infrastructure to technology start-up’s. Finally, we illustrate the employment of PMs as work-out loans especially in the ongoing sub-prime crisis.
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Figures

**Figure 1: Participating mortgage loans.** SIM stands for Shared Income Mortgage, SAM is Shared Appreciation Mortgage and SEM means Shared Equity Mortgage.
Figure 2: Dependence of the contractual interest rate $i$ on maturity $T$ of a Shared Income Mortgage (SIM). Parameters are $P_0 = 250\,000$, $K = 1\,000\,000$, $r = 0.1$, $\delta = 0.05$, $\sigma = 0.15$, $H_0 = 100\,000$, $L = 0.9$. Participation ratios are: $\theta_P \in \{1, 0.5, 0.25, 0.15, 0.1, 0.05\}$. 
Figure 3: Dependence of the participation ratio $\theta_p$ on maturity $T$ for a Shared Income Mortgage (SIM). Parameters are $P_0 = 250\,000$, $K = 1\,000\,000$, $r = 0.1$, $\delta = 0.05$, $\sigma = 0.15$, $H_0 = 100\,000$, $\bar{L} = 0.9$. Contractual interest rates are: $i \in \{0.095, 0.05, 0\}$. The dashed line represents the case $i = 0$. 


Figure 4: Real estate participation ratio $\theta^*_H$ (dashed line) and the profit participation ratio $\theta^*_P$ (solid line) for a Shared Equity Mortgage (SEM). Parameters $p_0 = 1\,000\,000$, $K = 1\,000\,000$, $r = 0.1$, $\delta = 0.05$, $\sigma = 0.15$, $\delta_H = 0.05$, $\sigma_H = 0.001$, $H = 100\,000$, $L = 0.9$. 

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