Portfolio and Consumption Decisions under Ambiguity for Regime Switching Mean Returns

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Abstract

This paper analyzes, in a continuous-time setting, dynamic consumption and portfolio decisions under ambiguity when the expected return is unobservable and follows a two-state Markov chain. Ambiguity is modeled by Chen and Epstein (2002)’s recursive multiple priors utility. Learning about investment opportunities introduces intertemporal hedging of ambiguity, which has the effect of mitigating hedging demand in addition to its impact on the mean-variance efficient allocation. Ambiguity can enrich the set of relation between the optimal portfolio allocation and the investment horizon. By calibrating the learning dynamics to the market data, we find that (1) moderate ambiguity aversion can generate optimal portfolio strategies that are nearly myopic; the optimal strategies with high ambiguity aversion are less sensitive to innovations in beliefs than the instantaneous mean-variance efficient strategies and (2) ambiguity reduces the welfare loss from ignoring learning.

JEL: G11; E21; D81; C61

Keywords: Ambiguity, Malliavin derivative, regime switching, portfolio choice

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1 Introduction

How should a long horizon investor make portfolio and consumption decisions when the time-varying investment opportunities are not directly observable? In the real world, investors cannot directly observe how investment opportunities evolve over time with perfect accuracy. Instead, they can learn about investment opportunities by observing market signals and then base their portfolio and consumption decisions on the inferred investment opportunities. In this setting, the inferred investment opportunity set is stochastic and the intertemporal hedging demand introduced by Merton (1971) becomes important to the dynamic portfolio decision. Further, the optimal portfolio depends on the investment horizon when the investor’s utility function is non-logarithmic.

The main contribution of this paper is to explicitly incorporate ambiguity (model uncertainty) into the analysis and study dynamic portfolio and consumption decisions in continuous time when an investor has incomplete information about the time-varying expected returns. We examine how ambiguity and parameter uncertainty affect intertemporal hedging demand, the optimal portfolio strategy and the optimal consumption-wealth ratio. Early works concerning the effect of ambiguity on dynamic consumption and portfolio decisions include Maenhout (2004), Uppal and Wang (2003) and Chen and Epstein (2002), but they confine themselves with constant and fully observable investment opportunities. Maenhout (2006) extends the analysis to a mean-reverting risk premium, and obtains closed-form solution for the optimal portfolio under ambiguity. Until now, few research has examined the implications of ambiguity for dynamic consumption and portfolio choice when investment opportunities are stochastic and non-linear, for instance, when they unobservable and regime switching. This paper attempts to fill this gap in line with Miao (2001), where a model setting is presented to analyze dynamic consumption and portfolio decisions under incomplete information and ambiguity. In this paper, we provide a more formal analysis and derive quantifiable implications of ambiguity.

In particular, we assume unobservable Markov switching model for investing environments. Then we

1 Under the assumptions of i.i.d. returns and isoelastic utility function, the seminal work of Merton (1969) derives the constant optimal portfolio in continuous time. Subsequent studies including Merton (1971), Campbell and Viceira (1999), Kim and Omberg (1996), Wachter (2002), Lynch (2001), Campbell et al. (2003), Schroder and Skiadas (1999), Detemple et al. (2003) and Liu (2007) consider the implications of time-varying investment opportunities for dynamic portfolio choice. A major finding is that return predictability can increase the optimal portfolio allocation for an investor with long horizons.

2 Liu (2008) further incorporates intermediate consumption decision, and solves the model explicitly using the martingale method by Cox and Huang (1989).

3 Skiadas and Schroder (2003) solve optimal portfolio and consumption policies for generalized recursive preferences, which incorporate recursive multiple priors utility as a special case, in stochastic environments. But they do not explicitly characterize the optimal controls for the investment opportunity set considered in this paper.
use Monte Carlo method based on Malliavin derivatives to compute the optimal portfolios and the consumption-wealth ratios. Finally, we provide economic explanations for numerical results. The method for deriving and computing the optimal portfolios closely follows Detemple et al. (2003).

We assume that stock prices are generated by a diffusion process whose drift depends on an unobservable regime variable of the economy. This regime variable follows a two-state continuous-time Markov chain. The investor makes the inference about the regime variable and learns about the “true” expected return through observing the current and the past stock prices. The continuous-time Markov switching model was first introduced by David (1997) to study asset pricing and portfolio choice. In the presence of the hidden state, parameter uncertainty leads to an additional risk, which is known as “estimation risk”. The standard approach dealing with learning about a hidden state is the Bayesian methodology. Previous studies for example, Dothan and Feldman (1986), Detemple (1986), Gennotte (1986), Feldman (1992), David (1997), Brennan (1998), Veronesi (2000), Brennan and Xia (2001), Xia (2001), Honda (2003), Ai (2006) and David (2008), assume that the conditional distribution of signals and the distribution of the hidden state are both correctly and uniquely specified given the past signals. Then these papers use recursive filtering methods to construct a stochastic process for the conditional estimates of the hidden state. In analyzing asset pricing and portfolio choice, they assume that the representative agent completely trusts the model for the conditional estimates without any concern about model uncertainty.

We introduce ambiguity and ambiguity aversion for the following three reasons. First, the Ellsberg Paradox and other related experimental evidence have demonstrated that the distinction between risk and ambiguity is behaviorally meaningful. Risk refers to the situation where a probability distribution can be precisely known, whereas ambiguity refers to the environment where information is too imprecise to be summarized by a single probability measure. According to Knight (1921) and Keynes (1936), ambiguity may be more important for economic decision-making. In making portfolio and consumption decisions, the investor only has a vague idea as to the true probability measure generated by the stochastic investment opportunity set. Instead of relying on a single subjective

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4 Lakner (1998) shows that if the mean return is unobservable and follows a mean-reverting Ornstein-Uhlenbeck process, then a closed-form solution for the optimal portfolio is available when an investor maximizes the expected utility from terminal wealth.

prior, the investor is uncertain about the priors relevant to her portfolio and consumption decisions. In this paper, we use a set of priors that is exogenously given and the recursive multiple priors utility\(^6\), which is developed in Chen and Epstein (2002), to capture the notions of ambiguity and ambiguity aversion. Ambiguity introduces distortions to the state dynamics, and ambiguity aversion imputes pessimism. Learning does not resolve ambiguity through time under the specification in Chen and Epstein (2002).

Second, incorporating ambiguity can also be motivated by the robust decision making, whose implications for finance and macroeconomics were first studied by Hansen and Sargent (2001) and Anderson et al. (2003). In particular, the investor worries that his model for the investment opportunities may be misspecified and wants to seek decision rules that are robust to model misspecification. Empirical evidence demonstrates that making inference about expected returns is very difficult and can be subject to substantial model misspecification. Merton (1980) documents the difficulty of estimating the mean return of stock prices and has argued that we need a very long period to efficiently estimate the expected return even when it is constant. This result is further strengthened and quantified by Goldenberg and Schmidt (1996). When the unobservable expected return is regime switching, it is reasonable to say that estimating the conditional distributions within each regime as well as the transition densities across different regimes becomes much more difficult. This implies substantial degree of model uncertainty. As a result, we introduce model uncertainty in addition to estimation risk. In particular, the investor is ambiguous about the joint stochastic processes for the return dynamics and the conditional estimates of the hidden state.

Third, for plausible degree of risk aversion, the optimal portfolio allocations assuming no ambiguity aversion are too high to be useful in guiding investment decisions. While it has been shown in Chen and Epstein (2002) that ambiguity can produce conservative portfolio strategies for i.i.d. returns, it is interesting and non-trivial to extend the result to the model with learning because ambiguity also exerts an impact on the dynamics of the intertemporal hedging demand.

Under incomplete information, ambiguity affects the optimal portfolio in two ways. On the one hand, as in the case of i.i.d. returns, ambiguity lowers the instantaneous mean-variance efficient stock allocation. On the other hand, intertemporal hedging of ambiguity contributes an additional component to hedging demand. This component is state-and-horizon dependent, and can either

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\(^6\) The discrete-time version of recursive multiple priors utility has been axiomatized in Epstein and Schneider (2003). See Gilboa and Schmeidler (1989) for the axiomatized multiple priors utility.
mitigate the optimal hedging demand or reverse its sign depending upon the degree of ambiguity aversion. In contrast to the model with only estimation risk, the optimal proportion of wealth allocated to stocks can either decrease or vary non-monotonically with the horizon. In regard to its effect on the optimal consumption-wealth ratio, ambiguity increases the ratio for low risk-averse investors while lowers the ratio for high risk-averse investors.

This paper contributes to a growing body of literature that examines the implications of ambiguity and robustness for portfolio choice and asset pricing. Several studies, for example Epstein and Wang (1994), Chen and Epstein (2002), Epstein and Miao (2003) and Cao et al. (2005) consider multiple priors utility framework, while other papers including Anderson et al. (2003), Cagetti, et al. (2002), Hansen and Sargent (2001), Hansen et al. (2006) and Hansen and Sargent (2007) examine the implications of robust decision-making for asset pricing. Epstein and Schneider (2007) assume multiple priors and multiple likelihoods, and allow beliefs to be updated prior by prior. They also study dynamic portfolio choice with learning under ambiguity. Leippold et al. (2007) further extend the model to continuous time, and study asset pricing implications. In this paper, we do not model learning under ambiguity when the underlying variable follows a Markov switching process. In other words, ambiguity is not endogenously generated from updating multiple priors when signals are observed. Instead, we allow agents to update beliefs in a Bayesian fashion but impose multiple priors on the Bayesian estimated model.

The rest of this paper is organized as follows: Section 2 presents the model and derives the optimal consumption and portfolio rules using the martingale method of Cox and Huang (1989) and Karatzas and Xue (1991); Section 3 describes data, estimation and model calibration; Section 4 discusses the results and gives economic interpretation; Section 5 concludes. The proofs and a discretization scheme of the Wonham filter are included in the Appendices.

2 The Model

2.1 Investment Opportunities and Information Structure

Time is continuous in the finite horizon $[0, T]$. Uncertainty is represented by a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t=0}^T, P)$ on which a standard Brownian motion $B$ and a two-state continuous-time Markov chain $\{\mu_t\}_{t=0}^T$ are defined. One risky asset and one riskless asset are available for
investment. The riskless asset price process $\beta$ follows

$$d\beta_t = \beta_t r dt,$$  \hspace{1cm} (2.1)

for a positive constant $r$. The risky asset price process $S$ satisfies

$$dS_t = S_t \mu dt + S_t \sigma dB_t, \quad S_0 > 0,$$ \hspace{1cm} (2.2)

where $\sigma$ is a constant. The market price of risk $\vartheta$ is defined as $(\mu - r)/\sigma$.

The state space is $\Xi = \{\mu_H, \mu_L\}$, where $\mu_H > \mu_L$, meaning that the state $H$ is the high-expected-return state. The infinitesimal generating matrix of $\{\mu_t\}_{t=0}^T$ is

$$\Lambda = \begin{bmatrix} -\lambda & \lambda \\ \chi & -\chi \end{bmatrix}$$

where $\lambda, \chi > 0$.\footnote{Honda (2003) assumes that a single parameter governs the transition densities, which implies that the transition probability of jumping from the low state to the high state is equal to that of falling from the high state to the low state.} The prior distribution\footnote{To avoid confusion, please notice the different meanings of “prior ” in “the prior of the two regimes ” and “multiple priors and/or the set of priors ”. The former refers to the statistical priors while the latter represents the agent’s subjective probability under which actions are evaluated.} of the two regimes is as such: at the initial time $t = 0$, the economy is in the high-expected-return state ($\mu_H$) with probability $\pi_0$ and in the low-expected-return state ($\mu_L$) with probability $1 - \pi_0$. Upon the arrival at state $i$ ($i = \mu_H$ or $\mu_L$), the process $\mu_t$ remains there with an exponentially distributed duration time, and then jumps to state $j$ ($j \neq i$). The transition probabilities over any interval of time can be constructed from the infinitesimal matrix $\Lambda$ via the exponential formula

$$T_s = \exp(s\Lambda).$$

The complete information filtration $\{\mathcal{F}_t\}$ is the augmented filtration generated by a standard Brownian motion $B$ and the regime switching mean returns, that is, $\mathcal{F}_t = \sigma(B_\tau, \mu_\tau, \tau \leq t)$. It is assumed that investors can observe neither the mean return $\mu$, nor the Brownian motion $B$. Rather, investors observes only the asset price process $S$. Let us denote by $\{\mathcal{F}_t^S\}$ the augmented filtration generated by the risky asset price process, where $\mathcal{F}_t^S = \sigma(S_\tau, \tau \leq t)$. Thus, investors’ information is represented...
by the filtration \( \{\mathcal{F}_t^S\} \) where each \( \mathcal{F}_t^S \subset \mathcal{F}_t \). Let us introduce the process \( \hat{B}_t \):

\[
\hat{B}_t = \int_0^t \frac{dS_r - S_r \hat{\mu}_r d\tau}{S_r \sigma}
\]

(2.3)

where \( \hat{\mu}_t \equiv \mathbb{E}[\mu_t | \mathcal{F}_t^S] \) is a measurable version of the conditional expectation of \( \mu \) with respect to the price filtration \( \{\mathcal{F}_t^S\} \). The Brownian motion \( \hat{B} \) is often referred to as an innovation process in the standard literature (Liptser and Shiryaev 2001). The following lemma is common and crucial in the literature on dynamic models with learning in continuous time.

**Lemma 1** \( \hat{B} \) is a \( (P, \{\mathcal{F}_t^S\}) \)-Brownian motion. Moreover, the augmented filtration generated by the Brownian motion \( \hat{B} \) coincides with \( \{\mathcal{F}_t^S\} \).

**Proof.** See Liptser and Shiryaev (2001). □

Define the filtered probability \( \pi_t \) as the posterior probability that the current regime is in the high mean return state, that is

\[
\pi_t = \mathbb{P}(\mu_t = \mu_H | \mathcal{F}_t^S) \quad \pi_0 \text{ is given}
\]

According to Theorem 9.1 in Liptser and Shiryaev (2001), \( \pi_t \) follows the stochastic differential equation (SDE)

\[
d\pi_t = [\lambda - (\lambda + \chi)\pi_t]dt + \pi_t(1 - \pi_t)\frac{(\mu_H - \mu_L)}{\sigma^2}d\hat{B}_t
\]

(2.4)

which is a non-linear filtering equation. The drift and the volatility coefficients satisfy the Growth and Lipschitz conditions. It is obvious that the precision of the conditional estimates of the hidden state fluctuates in a stochastic way. This is in contrast to the continuous-time Kalman filter, which has been considered in Gennotte (1986) and Brennan (1998), where the conditional variance process is deterministic, and the precision of the conditional estimates converges to a steady state. In (2.4), \( \pi_t \) determines both the conditional mean estimate and the precision. To explore more properties of the filter equation (2.4), it is useful to rewrite it in terms of the original standard Brownian motion \( B_t^{\uparrow} \). Suppose the investor knows that during some time interval \([t_1, t_2]\) the true mean returns is \( \mu_t \), then the process \( \pi_t \) can be described by the following SDE

\[
d\pi_t = \left[ \lambda - (\lambda + \chi)\pi_t + \frac{\pi_t(\mu_H - \hat{\mu}_t)(\mu_t - \hat{\mu}_t)}{\sigma^2} \right] dt + \frac{\pi_t(\mu_H - \hat{\mu}_t)}{\sigma} dB_t
\]

See David (1997) and Veronesi (2000) for more discussion.
During time interval \([t_1, t_2]\), if the true mean return \(\mu_t = \mu_H\), and zero otherwise \((\mu_t = \mu_L)\), the second component in the drift tends to pull \(\pi_t\) toward one. Given that \(\mu_t = \mu_H\), the second term in the drift equals \(\pi_t(\mu_H - \mu_t)^2/\sigma^2\), which tends to drive \(\pi_t\) to 1 over time. When \(\pi_t\) approaches 1, \(\pi_t(\mu_H - \mu_t)^2/\sigma^2\) and the diffusion term converge to zero, which makes the first term, \(\lambda - (\lambda + \chi)\pi_t\) play a dominant role. This term prevents \(\pi_t\) from converging to 1 and tends to drive \(\pi_t\) inside the interval.

The instantaneous standard deviation term, \(\pi_t(1 - \pi_t)(\mu_H - \mu_L)/\sigma\) determines estimation risk. When the investor is less confident about the current regime (\(\pi_t\) takes a value near 1/2), the diffusion term in (2.4) contributes more to the updating rule and the investor relies more on the innovation in the information. When the investor is fairly confident about the current regime (\(\pi_t\) approaches the boundary of the interval \([0, 1]\)), then the diffusion term is less significant. With larger difference between the high regime \((\mu_H)\) and the low regime \((\mu_L)\), more information can be revealed by the innovation, and the investor puts more weight on the new information. In addition, higher return volatility \(\sigma\) implies more noisy signals. Thus, new information is less useful in updating beliefs and the scope of estimation risk is limited.

### 2.2 Optimal Consumption and Portfolio Choice

Denote the wealth process by \(\{X_t\}_{t=0}^T\). A portfolio choice \(\psi\) is a \(\{\mathcal{F}_t^S\}\)-adapted progressively measurable process such that \(\int_0^T |\psi_s|^2 \, ds < \infty\). \(\psi_t\) represents the proportion of total wealth invested into the risky asset at time \(t\). As a result, \(1 - \psi_t\) is the proportion invested into the riskless asset. Given the initial wealth endowment \(X_0\), the investor makes consumption and investment decisions at time \(t\) based on information contained in \(\{\mathcal{F}_t^S\}\). The budget constraint is given by

\[
    dX_t = \{[r + \psi_t(\mu_t - r)]X_t - c_t\}dt + X_t\psi_t\sigma dB_t
\]

which can be rewritten as

\[
    dX_t = X_t[\psi_t \frac{dS_t}{S_t} + (1 - \psi_t)r dt] - c_t dt
\]

In general, the following procedure is often employed to solve stochastic dynamic optimization problems under incomplete information\(^{10}\).

Two-step Procedure: The investor solves the optimal portfolio and consumption decision in two stages: (1) the investor solves the (non-linear)filtering problem and derives conditional estimates; (2) the investor makes portfolio and consumption decisions based on the conditional estimates.

However, since the investor may cast doubt on the non-linear filter generated in Step (1), we modify the standard two-step procedure to a three-step procedure:

Three-step Procedure: The investor solves the optimal portfolio and consumption decision in three stages: (1) the investor solves the (non-linear)filtering problem and derives conditional estimates; (2) the investor doubts the conditional estimates and possesses multiple priors; (3) given the set of priors, the investor makes portfolio and consumption decisions with recursive multiple priors utility.

In Markov switching model, the conditional expected return $\hat{\mu}_t$ is

$$\hat{\mu}_t = \mu_H \pi_t + \mu_L (1 - \pi_t).$$

Since $\pi_t$ satisfies stochastic differential equation (2.4), by Ito’s lemma, $\hat{\mu}$ satisfies

$$d\hat{\mu}_t = [\lambda (\mu_H - \mu_L) - (\lambda + \chi)(\hat{\mu}_t - \mu_L)]dt + \frac{1}{\sigma} (\hat{\mu}_t - \mu_L)(\mu_H - \hat{\mu}_t) d\hat{B}_t$$

which characterizes the dynamics of the conditional expected returns. By (2.2) and (2.3), the perceived return process is

$$dS_t = S_t \hat{\mu}_t dt + S_t \sigma d\hat{B}_t$$

and the budget constraint (2.5) becomes

$$dX_t = \{[r + \psi_t (\hat{\mu}_t - r)]X_t - c_t\} dt + X_t \psi_t \sigma d\hat{B}_t.$$  \hspace{1cm} (2.7)

Now the investor’s optimization problem has been converted to one under complete information, where the state processes are (2.4) and (2.7). The innovation process $\hat{B}$ drives uncertainty and the investor’s information filtration is $\mathcal{F}_t^\hat{B}$. 
2.2.1 Recursive Multiple Priors Utility

The investor’s utility is defined on a terminal wealth $X_T$ (a non-negative random variable which is $\mathcal{F}_T^S$-measurable) and a consumption process. Suppose that the consumption process $c$ is non-negative, progressively measurable with respect to the filtration $\{\mathcal{F}_t^S\}$ and square integrable with $E[\int_0^T c_t^2 dt] < \infty$. Also assume that terminal wealth satisfies $E(|X_T|^2) < \infty$. The recursive multiple priors utility process is defined on the set of priors $\mathcal{P}$, which is constructed through $\{\mathcal{F}_t^S\}$-adapted density generators $\theta = (\theta_t) \in \Theta$ satisfying $\sup |\theta_t| \leq \kappa$, where $\kappa \geq 0$. According to Chen and Epstein (2002), this specification is referred to as $\kappa$-ignorance. $\kappa$ can also be interpreted as an ambiguity aversion parameter. Each density generator $\theta$ delivers a $(P, \{\mathcal{F}_t^S\})$-martingale $(z^\theta_t)\quad z^\theta_t = \exp \left( -\frac{1}{2} \int_0^t |\theta_s|^2 ds - \int_0^t \theta_s d\hat{B}_s \right) , \quad 0 \leq t \leq T. \quad (2.8)

Then the set of priors is constructed as

$$\mathcal{P} = \left\{ Q^\theta : \theta \in \Theta, \frac{dQ^\theta}{dP} = z^\theta_T \right\}$$

Because $z^\theta_t$ is a martingale, we also have

$$\frac{dQ^\theta}{dP} \bigg|_{\mathcal{F}_t^S} = z^\theta_t.$$

The investor is ambiguous about whether $(\hat{B}_t)$ is a Brownian motion with respect to the investor’s filtration. Because the filtered probabilities $\pi$ constitute the investor’s information set, ambiguity can be interpreted as uncertainty about the evolution of the filtered probabilities. Girsanov’s Theorem implies that $\hat{B}_Q^t \equiv \hat{B}_t + \int_0^t \theta_s ds$ is a Brownian motion under the probability measure $Q$. The multiplicity of the set of priors captures the investor’s doubt on the true model.

For each measure $Q$ in the set of priors, the utility process for $c$ is denoted by $V^Q_t$, which is defined as

$$V^Q_t = E_Q \left[ \int_t^T e^{-\rho(s-t)} u(c_s) ds + e^{-\rho(T-t)} u(X_T) \bigg| \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

\[\text{We can think a scenario similar to the one given by Chen and Epstein (2002): there is sequence of ambiguous Ellsberg urns, each containing a filtered probability sampled in discrete time (the instantaneous volatility of the filtered probabilities is non-zero since they cannot be precisely estimated). If the nthe draw is from the the n urn, ambiguity persists in a generic way. The increments } (d\hat{B}_t) \text{ of the innovation process } \hat{B} \text{ can be taken as the counterparts of the Ellsberg urns.}\]
where $u(\cdot)$ is the utility function and $\rho > 0$ is the discount rate.

Let $f(c, y, z)$ denote the aggregator

$$f(c, y, z) = u(c) - \rho y - \theta z$$

where $z$ is a positive scalar. Then Girsanov Theorem and dynamic consistency imply that $(V^Q_t)$ solves the backward stochastic differential equation (BSDE):

$$dV^Q_t = -f(c_t, V^Q_t, \sigma^Q_t)dt + \sigma^Q_t dB_t, \quad V^Q_T = u(X_T).$$

Under certain technical conditions presented in El Karoui et al. (2001), this BSDE has a unique solution $(V^Q, \sigma^Q)$.

The recursive multiple priors utility process $V_t(c, X_T)$ is defined as:

$$V_t(c, X_T) = \min_{Q \in \mathcal{P}} E^Q \left[ \int_t^T e^{-\rho(s-t)} u(c_s) ds + e^{-\rho(T-t)} u(X_T) | \mathcal{F}_t \right]$$

where the volatility term $\sigma^V_t$ is endogenous and is part of the complete solution to the BSDE. Then

$$\max_{\theta \in \Theta} \theta_t \times \sigma^V_t = \theta^*_t \times \sigma^V_t, \quad \text{where } \theta^*_t = \kappa \times \text{sgn}(\sigma^V_t)$$

**Theorem 2** The utility process $\{V_t(c, X_T)\}$ is dynamically consistent and satisfies the BSDE:

$$dV_t = [-u(c_t) + \rho V_t + \max_{\theta \in \Theta} \theta_t \times \sigma^V_t] dt + \sigma^V_t dB_t, \quad V_T = u(X_T).$$

where $\text{sgn}(x)$ is the $d$-dimensional vector with $i$th component equal to $\frac{|x_i|}{x_i}$ if $x_i \neq 0$ and $= 0$ otherwise. For any $d$-dimensional vector $y$, $y \otimes \text{sgn}(x)$ denotes the vector with $i$th component $y_i \times \text{sgn}(x_i)$.

**Proof.** See the proof of Theorem 2.2 in Chen and Epstein (2002). Notice that adding a bequest function is of no consequence for the main arguments in the proof if concavity and increasing monotonicity can hold with respect to terminal wealth. □

According to (2.10), if $\sigma^V_t > 0$, $\theta^*_t = \kappa$; and if $\sigma^V_t < 0$, $\theta^*_t = -\kappa$. In the next part, we characterize
two conditions, respectively, to deliver these two results.

Assume that the agent has constant relative risk aversion (CRRA) instantaneous utility function $u$,

$$u(c) = \begin{cases} 
c^{1-\gamma} / (1-\gamma) & \gamma \neq 1 
\log(c) & \gamma = 1
\end{cases}$$

where $\gamma$ is the coefficient of relative risk aversion.

### 2.2.2 The Martingale Formulation

This part employs the martingale method to solve the investor's dynamic optimization problem with multiple priors. The martingale method was first developed by Cox and Huang and further extended to incomplete information economies by Karatzas and Xue (1991) and Lakner (1998).

The perceived market price of risk $\hat{\vartheta}_t$ is a measurable version of the conditional expectation of $\vartheta$ with respect to $\{\mathcal{F}^S_t\}$, and is given by

$$\hat{\vartheta}_t = \frac{\hat{\mu}_t - r}{\sigma}$$

(2.11)

When the Novikov’s condition holds, that is,

$$E\left(\exp\left\{\frac{1}{2} \int_0^T \hat{\vartheta}_t^2 dt\right\}\right) < \infty,$$

and markets are complete with respect to filtration $\{\mathcal{F}^S_t\}$, the equivalent martingale measure exists and is uniquely determined. Define the $(P, \{\mathcal{F}^S_t\})$-martingale $\{\zeta_t\}$ by

$$\zeta_t = \exp\left(-\int_0^t \hat{\vartheta}_t d\hat{B}_\tau - \frac{1}{2} \int_0^t \hat{\vartheta}_t^2 d\tau\right).$$

(2.12)

$\zeta_t$ satisfies $d\zeta_t = -\zeta_t \hat{\vartheta}_t d\hat{B}_t$ with $\zeta_0 = 1$. By Girsanov’s Theorem, the process $\tilde{B}$ defined by

$$\tilde{B}_t = \hat{B}_t + \int_0^t \hat{\vartheta}_\tau d\tau$$

(2.13)

is a $(\tilde{P}, \{\mathcal{F}^S_t\})$-Brownian motion, where $\tilde{P}$ is a probability measure on $(\Omega, \{\mathcal{F}^S_t\})$ defined by Radon-Nikodym derivative $d\tilde{P} / dP = \zeta_T$. In complete markets, the martingale method can be used to solve for the optimal portfolio and consumption decision rules. The investor’s stochastic dynamic optimization
problem is converted into a static Arrow-Debreu problem to solve for the optimal consumption and terminal wealth. Then the optimal portfolio is retrieved to finance the intermediate consumption and attains the terminal wealth. Define the state price deflator process $\xi$ by $\xi_t = e^{-rt}\zeta_t$.

Under $\kappa$–ignorance specification, the static variational problem is given by

$$\sup_{c, X_T} V_0(c, X_T)$$

subject to the budget constraint

$$E \left[ \int_0^T \xi_t c_t + \xi_T X_T \right] \leq X_0$$

where $V_0(c, X_T)$ is the recursive multiple priors utility. The supergradient for $V$ at the consumption process $c$ is a process $(p_t)$ satisfying

$$V(c', X_T) - V(c, X_T) \leq E \left[ \int_0^T p_t(c'_t - c_t) dt \right]$$

for all $c$ and $c'$ in consumption space. The supergradient for $V$ at terminal wealth $X_T$ is a random variable $(p_T)$ such that

$$V(c, X_T') - V(c, X_T) \leq E \left[ p_T(X_T' - X_T) \right]$$

It can be shown that the supergradients are

$$p_t(c) = e^{-\rho t} u'(c_t) z_t^\theta.$$  

and

$$p_T(X_T) = e^{-\rho T} u'(X_T) z_T^\theta.$$  

First order conditions can be expressed in terms of the utility supergradients. (see proof of Theorem 3, Appendices) The following theorem gives explicit formulae to the optimal consumption ($c^*$), terminal wealth $X_T^*$, and the optimal portfolio allocation ($\psi^*$). The optimal portfolio can be characterized in terms of stochastic integrals and the Malliavin derivative of the state belief, where the dynamics of the Malliavin derivatives follows some SDE.

13 See Chen and Epstein (2002) for the definition of supergradient.
Theorem 3 Let

\[ G_t = \hat{E}_t \left[ \int_t^T e^{-rs} e^s \left( \int_t^s D_t \hat{\vartheta}(d\hat{B}_\tau + \hat{\vartheta}_\tau d\tau) \right) ds + e^{-rT} X^*_t \int_t^T D_t \hat{\vartheta}_s (d\hat{B}_s + \hat{\vartheta}_s ds) \right] \]

where \( \hat{E}_t \) is the conditional expectation operator under \( \hat{P} \).

(a) If the following condition holds,

\[ (\hat{\vartheta}_t - \kappa) + \frac{e^{rt}}{X^*_t} G_t > 0 \]

then \( \theta^* = \kappa \).

If

\[ (\hat{\vartheta}_t + \kappa) + \frac{e^{rt}}{X^*_t} G_t < 0 \]

then \( \theta^* = -\kappa \).

(b) The optimal consumption process is given by

\[ c^*_t = \left( \frac{e^{-\rho t} \hat{\vartheta}^*}{y \xi_t} \right)^{\frac{1}{\gamma}} \] (2.15)

The optimal terminal wealth is given by

\[ X^*_T = \left( \frac{e^{-\rho T} \hat{\vartheta}^*_T}{y \xi_T} \right)^{\frac{1}{\gamma}} \] (2.16)

where \( y > 0 \) is the Lagrange multiplier and satisfies

\[ y = \left( E \left[ \int_0^T (\xi_t)^{\frac{\gamma-1}{\gamma}} (e^{-\rho t} z^*_t)^{\frac{1}{\gamma}} dt + (\xi_T)^{\frac{\gamma-1}{\gamma}} (e^{-\rho T} z^*_T)^{\frac{1}{\gamma}} \right] / X_0 \right)^{\gamma} \] (2.17)

(c) The optimal portfolio is given by

\[ \psi^*_t = \frac{\hat{\mu}_t - \gamma}{\gamma \sigma^2} - \frac{\theta^*}{\gamma \sigma} + \frac{1 - \gamma}{\gamma} \frac{e^{rt}}{\sigma X^*_t} G_t \] (2.18)

(d) The random variable \( D_t \hat{\vartheta}_\tau \) is the Malliavin derivative, in which \( D_t \hat{\mu}_s \) satisfies the following SDE:

\[ d(D_t \hat{\mu}_s) = -(\lambda + \chi) D_t \hat{\mu}_s ds + \left( \frac{1}{\sigma} (\mu_H + \mu_L - 2\hat{\mu}_s) d\hat{B}_s \right) D_t \hat{\mu}_s \] (2.19)
subject to the boundary condition \( \lim_{s \to t} D_t \hat{\mu}_s = \frac{(\hat{\mu}_t - \mu_L)(\mu_H - \hat{\mu}_t)}{\sigma} \).

**Proof.** see Appendices.

Also let
\[
\bar{G}_t = \hat{E}_t \left[ \int_t^T e^{-rs} \bar{c}_s^* \left( \int_t^s D_t \bar{\theta}_r (d\hat{B}_r + \hat{\vartheta}_r d\tau) \right) ds + e^{-rT} \bar{X}_T^* \int_t^T D_t \hat{\vartheta}_s (d\hat{B}_s + \hat{\vartheta}_s ds) \right]
\]
where \( \bar{c}_s^* \) and \( \bar{X}_T^* \) are the optimal consumption and terminal wealth obtained in the expected utility model, and are given by
\[
\bar{c}_t^* = \left( \frac{e^{-\rho t}}{y \xi_t} \right)^{\frac{1}{\gamma}} \quad \text{and} \quad \bar{X}_T^* = \left( \frac{e^{-\rho T}}{y \xi_T} \right)^{\frac{1}{\gamma}}
\]
respectively. Then (2.18) can be rewritten as
\[
\psi_t^* = \frac{\hat{\mu}_t - \bar{c}_t^*}{\gamma \sigma^2} - \frac{\kappa}{\gamma \sigma^2} + \frac{1 - \gamma}{\gamma} \frac{e^{\gamma t}}{\sigma \bar{X}_t^*} \bar{G}_t + \frac{1 - \gamma}{\gamma} \frac{e^{\gamma t}}{\sigma \bar{X}_t^*} (G_t - \bar{G}_t) \quad (2.20)
\]
The first term is usually refer to as myopic demand, which is instantaneously mean-variance efficient.
The second term shows the effect of ambiguity on the myopic demand. The sum of the first two terms can be named as *ambiguity-adjusted myopic demand*. The third term arises because of estimation risk, and represents the hedging demand against future changes in the state dynamics of conditional expected returns. The fourth term appears due to intertemporal hedging of ambiguity, which depends on the state dynamics as well as the investment horizon. Under incomplete information, ambiguity not only influences the myopic portfolio allocation but also exerts impact on hedging demand through dynamic learning. In the expressions for the optimal portfolio, the random variable \( D_t Y_s \) captures the effect of an innovation in the Brownian motion \( B \) at time \( t \) on the state variable \( Y \) at time \( s \). It reflects the response of a variable \( Y_s \) to a past uncertainty shock at time \( t \).

The following remarks address several special cases of Theorem 3:

**Remark 4** When returns are i.i.d. and the investor is ambiguity-averse, only the first two terms arise. See Chen and Epstein (2002) for the proof. A similar result is obtained in Maenhout (2004) in which ambiguity is modeled by the robust control approach. While Maenhout (2004) shows that the effect of ambiguity or robustness on the optimal portfolio allocation is of the second order (enlarging robustness is observationally equivalent to increasing effective risk aversion), the effect of ambiguity is of the first order in the multiple priors model considered here (an increase in ambiguity leads to a
lower market price of risk perceived by the investor).

Remark 5 When $\kappa = 0$, the second term and the fourth term vanish; the investor only hedges for estimation risk. See Brennan (1998) and Honda (2003) for related examples.

Remark 6 When $\gamma = 1$, the investor behaves myopically in the sense that she does not hedge intertemporally for the stochastic variations of investment opportunities. However, the logarithmic investor does take into account ambiguity in making portfolio decision. If $\kappa = 0$, we can obtain the result in Feldman (1992).

3 Data, Estimation and Model Calibration

3.1 Data and estimation

The parameters of the joint stochastic processes for the stock returns and the two-state continuous time Markov switching process are estimated using the annual U.S. stock market returns. The stock return is the return on the value-weighted index of NYSE stocks and is obtained from the CRSP monthly returns file VWRETD for the period from January 1942 through December 2006. The inflation rate is calculated using FRED St.Louis CPI data file. The real riskless interest rate $r$ is estimated to be approximately 3.4 percent adjusted by the annual inflation rate. The real annual market returns are obtained by suitably compounding the nominal monthly returns and then adjusting for inflation rate. Then we estimate the discrete time analogue of the continuous-time Markov switching model using an EM algorithm to compute maximum-likelihood estimates, as described in Hamilton (1989). The infinitesimal generating matrix $\Lambda$ is estimated using the method in Israel et al. (2001):

$$\Lambda = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} (P - I)^i$$

where $P$ is the transition probability matrix, $I$ is the identity matrix, and the series is approximated with length 10. Table 1 presents the parameter estimates. It is obvious that the high-regime mean return significantly dominates the low-regime mean return in terms of time duration.

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14 The discrete-time version is

$$S_{t+1} = S_t \cdot e^{(\mu_1 - \frac{1}{2} \sigma^2) \Delta t + \sigma \epsilon_{t+1}}, \quad \epsilon_{t+1} \sim N(0, 1)$$

where $\mu_1$ follows a two-state Markov Chain and $\Delta t$ is set to 1.

15 This method has also been used in David (2008) to estimate the generator.
3.2 Monte Carlo simulation

We use the Malliavin derivative based Monte Carlo (MDMC) method in Detemple et al. (2003) to compute the optimal portfolio strategy. Since the martingale method and the Clark-Ocone formula allow us to explicitly derive the optimal portfolio rule, we do not need to employ some numerical optimization algorithms to implement stochastic dynamic programming. Simulating the optimal controls that are in closed-form can improve the efficiency of numerical approximations and the accuracy of numerical results.

We run the random number generator to simulate a large number of sample paths for the relevant state variables and numerically evaluate the stochastic integrals in the expressions for the optimal portfolio and the consumption-wealth ratio. The Malliavin derivative and other state variables are simulated using a variance stabilizing technique, as suggested in Detemple et al. (2003). The number of Monte Carlo replications is 20,000. C++ is the programming language. The initial wealth is assumed to be $1, so that the consumption is equal to the consumption-wealth ratio.

3.3 Detection error probabilities

Anderson et al. (2003) relates the calibration of model uncertainty to a model selection problem in a Bayesian context. Maenhout (2004) examines the effects of model uncertainty on portfolio choice and asset pricing using detection error probabilities in an i.i.d. context. In a subsequent work, Maenhout (2006) provides a new scheme to compute detection error probabilities when the approximating model features time variation and mean-reverting. For constant ambiguity aversion, it is straightforward to use the detection error probabilities technique to calibrate ambiguity.

When choosing between two potential data generating processes, a decision maker performs likelihood ratio tests under both models given available data. The two models are difficult to distinguish if the probability of mistakenly rejecting one model in favor of the other is high. This probability is given by the probability that the log-likelihood ratio is negative when the rejected model is the true data generating process. In continuous time model, the log-likelihood ratio is given by the log of the Radon-Nikodym derivative. Specifically, the log of the Radon-Nikodym derivative of the distorted probability measure $Q$ with respect to the reference probability measure $P$ is

$$\eta_{1,t} \equiv \log\left( \frac{dQ}{dP} \right|_{\mathcal{F}_t^S} ) = - \int_0^t \theta_s^* dB_s - \frac{1}{2} \int_0^t (\theta_s^*)^2 ds$$
The log of the Radon-Nikodym derivative of the probability measure $P$ with respect to the probability measure $Q$ is

$$
\eta_{2,t} \equiv \log\left(\frac{dP}{dQ} \mid \mathcal{F}_t^S\right) = \int_0^t \theta^*_s d\hat{B}_s + \frac{1}{2} \int_0^t (\theta^*_s)^2 ds
$$

If model $P$ is true, the decision maker will mistakenly reject it in favor of model $Q$ based on a finite sample with size $N$ when $\eta_{1,N} > 0$. Conversely, if model $Q$ is the correct model, it will be rejected erroneously when $\eta_{2,N} > 0$. Assuming the initial prior of 0.5 on each model, the detection error probability $\varepsilon_N(\theta)$ based on a sample size $N$ is defined as

$$
\varepsilon_N(\theta) = 0.5 \Pr(\eta_{1,N} > 0 \mid P) + 0.5 \Pr(\eta_{2,N} > 0 \mid Q).
$$

The detection error probability depends on $\theta$ in that as $\theta$ increases, models $P$ and $Q$ are easier to distinguished statistically from each other and the detection error probability shrinks. For the case of $\kappa$—ignorance, it is straightforward to derive that $\varepsilon_N(\kappa)$ is given by

$$
\varepsilon_N(\kappa) = \Pr\left( Z < -\frac{\kappa}{2} \sqrt{N} \right)
$$

where $Z$ is from the standard normal distribution.\[16\] The detection error probabilities for the sample in this paper are 5%, 10%, 20% and 30%, corresponding to the ambiguity aversions $\theta^*(\kappa) = 0.409, 0.317, 0.208$ and 0.129 respectively. Anderson et al. (2003) advocates 10% as the lower bound of detection error probabilities.

4 Results and Discussion

This section presents simulation results and provides economic insight to understand the results. Section 4.1 discusses the intertemporal hedging behavior and relates it to the consumption-wealth ratio. Section 4.2 studies the effects of estimation risk and ambiguity on hedging demand. Section 4.3 calibrates the learning dynamics to the monthly and the daily market returns, and examines the effect of ambiguity on the welfare loss from ignoring learning.

\[16\] It can be shown that this results also holds for constant ambiguity aversion $-\kappa$, but not for ambiguity aversion switching between $\kappa$ and $-\kappa$. A main reason for the equilibrium ambiguity aversion not changing its sign is that the utility, in most cases, is monotonic in the state variable.
4.1 Hedging Demand and the Consumption-Wealth Ratio

**Result 1** Assuming no ambiguity, when the market price of risk is positive, learning reduces the optimal demand for stocks for $\gamma > 1$, while increases the optimal demand for $\gamma < 1$. That is, if $\kappa = 0$ and $\hat{\vartheta}_t > 0$, learning induces negative hedging demand when $\gamma > 1$, and positive hedging demand when $\gamma < 1$.

The intertemporal hedging demand is induced to hedge variations in the investment opportunity set. Result 1 can be explained by the interaction between the income effect and the substitution effect, as pointed out by Campbell and Viceira (1999) and Honda (2003), among others. As state beliefs imply higher conditional expected returns, the income effect increases the current consumption due to higher purchasing power, while the substitution effect tends to decrease the current consumption since the investor is more willing to increase investment in stocks. When $\gamma > 1$, the income effect outweighs the substitution effect and the consumption-wealth ratio increases with more optimistic state beliefs. When $\gamma < 1$, the substitution effect dominates and the consumption-wealth ratio decreases with more favorable investment opportunities. In order to make consumption stable across different state beliefs, an investor with $\gamma > 1$ wants to choose a portfolio that can deliver more wealth when investment opportunities are unfavorable, whereas an investor with $\gamma < 1$ prefers a portfolio that can generate more wealth when investment opportunities are good. Learning creates a perfect positive correlation between innovations to the stock returns and innovations to the conditional expected returns, as can be seen from SDEs (2.4) and (2.13). Thus, the investor with $\gamma > 1$ will optimally hold less stocks relative to the myopic allocation and hedging demand is negative, but the opposite is true for the investor with $\gamma < 1$.

**Result 2** Ambiguity reduces the myopic stock allocation and mitigates hedging demand when the perceived market price of risk is positive, that is, when $\hat{\vartheta}_t - \kappa > 0$.

**Result 3** The consumption-wealth ratio is increasing in $\hat{\vartheta}_t$ but decreasing in $\kappa$ when $\hat{\vartheta}_t - \kappa > 0$ and $\gamma > 1$.

To understand these two results, we notice that ambiguity imputes two distortions, respectively, to the return process and the dynamics of the conditional expected returns. The first distortion has the effect of lowering the myopic stock allocation, and the second distortion affects hedging demand. Under incomplete information, learning creates an additional channel for ambiguity to
affect the optimal portfolio. When $\hat{\theta}_t - \kappa > 0$, the optimal ambiguity aversion parameter $\theta^*$ is equal to $\kappa$, and the distorted conditional beliefs feature pessimistic changes. For the investor with $\gamma > (\gamma <) 1$, an increase in ambiguity reduces (increases) the consumption-wealth ratio because the income (substitution) effect dominates. As a result, when $\gamma > 1$, an optimistic change in state beliefs under ambiguity will lead to a smaller increment in the consumption-wealth ratio than an optimistic change without ambiguity. For moderate degree of ambiguity, hedging demand is mitigated because the incentive of stabilizing consumption becomes less under ambiguity. In the presence of very high degree of ambiguity, the consumption-wealth ratio may even decrease with more optimistic state beliefs, which will change the sign of hedging demand. When $\gamma < 1$, the substitution effect dominates and the consumption-wealth ratio rises with ambiguity. Hedging demand decreases as a result. Figure 1 plots the optimal portfolio allocation and hedging demand for different risk aversions and ambiguity aversions assuming that $\pi$ is equal to its steady state value.

For an expected utility investor, if the conditional expected return becomes sufficiently negative, a decline in the belief of the high regime represents a more favorable change in the investment opportunities since the investor can engage in short selling to exploit profits. Thus, the sign of hedging demand is reversed for both cases of $\gamma > 1$ and $\gamma < 1$. Nevertheless, the impact of ambiguity deserves more discussion. When a state belief implies a significantly negative expected return, an ambiguity-averse investor will behave in a conservative way in that she will adjust the conditional estimate of the high-regime upward. This occurs because ambiguity implies lack of confidence about the profitable short-selling opportunities and $\theta^* = -\kappa$. Three consequences then arise: (1) the optimal stock allocation decreases (in absolute value), which implies a less aggressive short-selling strategy; (2) intertemporal hedging of ambiguity can decrease hedging demand or reverse its sign depending on the degree of ambiguity aversion; and (3) when $\gamma > (\gamma <) 1$, the consumption wealth ratio becomes higher (lower) under ambiguity. These results are shown in Table 2 and Figure 2, where Table 2 shows the case of $\pi_0 = 0.1$ and Figure 2 shows the scenario when $\pi_0$ ranges from 0.02 to 0.2.

From Table 2, we also notice that the consumption-wealth ratio is non-monotonic in $\gamma$. This happens because risk aversion and the elasticity of intertemporal substitution are inversely related, as explained in Campbell and Viceira (1999). In addition, we find that when $\gamma > 1$, hedging demand is non-monotonic in $\gamma$ and attains maximum (in absolute value) when $\gamma$ is approximately equal to 2.
4.2 Estimation Risk, Ambiguity and Hedging Demand

For an ambiguity-averse investor, both estimation risk and ambiguity can affect hedging demand. As is shown in (2.24), we can decompose the hedging portfolio into two components: the hedging demand for estimation risk and the hedging demand for ambiguity. The distortion caused by ambiguity to the dynamics of state beliefs is proportional to the standard deviation of the posterior beliefs $\pi_t$. This illustrates that because high estimation risk implies more difficulty in estimating the expected return, so that the investor believes that the “true” estimates deviate further from the Bayesian estimates. Thus, high estimation risk can cause high hedging demand for ambiguity. The effects of ambiguity on hedging demand, the optimal stock allocation, the ratio of hedging demand to the optimal demand for stocks and on the consumption-wealth ratio are shown in Figure 3 and Figure 4, where the simulation results are presented for $\gamma = 4$ and $\gamma = 8$.\(^{17}\) When $\pi$ is near the boundary of the interval $[0,1]$, the investor is almost sure about the current regime and estimation risk is low. On the one hand, the hedging demand for estimation risk is small. On the other hand, since low estimation risk implies high precision of the estimates, the distortion caused by ambiguity to the process of the estimates is small and the effect of ambiguity on hedging demand is also small. When $\pi$ takes values near around 0.6, the investor is far less sure about the current regime and estimation risk is high, leading to a large distortion by ambiguity to the dynamics of state beliefs. In this case, both the hedging demand for estimation risk and the hedging demand for ambiguity are high. In addition, although ambiguity reduces the myopic stock allocation and mitigates hedging demand, its effect on the ratio of hedging demand to the optimal demand for stock is ambiguous, as shown in Figure 3 and Figure 4.

Table 3 presents the simulation results for different low-regime mean returns. The low(high)-regime mean return can affect the optimal stock allocation in two ways. When the low-regime mean return declines, the myopic allocation is reduced. In addition, when there is a large difference between the high regime and the low regime, it is highly probable that a piece of new information can reveal much about the current state. In this case, the investor will put more weight on the new information when updating her beliefs, which will lead to high estimation risk and high hedging demand. It is worthwhile to note that although a lower low-regime mean return and an increase in ambiguity both can decrease the consumption-wealth ratio, their effects on hedging demand are qualitatively different since higher ambiguity reduces hedging demand.

\(^{17}\)In Figure 3 and Figure 4, we consider positive risk premium only and reset the low-regime mean return to 0.0618.
Table 4 shows the effect of return volatility on the myopic allocation and hedging demand. It is obvious from (2.4) and (2.13) that a lower return volatility is associated with a smaller distortion by ambiguity to the return dynamics, however, a larger distortion by ambiguity to the dynamics of the conditional estimates. When the return volatility declines, signals are less noisy and more weight will be put on the new information in updating beliefs. Thus, a low \( \sigma \) implies high estimation risk, which causes high hedging demand for estimation risk and also high hedging demand for ambiguity.

The transition density parameters \( \lambda \) and \( \chi \) are the key determinants of mean reversion in the dynamics of state beliefs. \( \chi \) plays the role of preventing \( \pi_t \) from converging to 1 while \( \lambda \) tends to refrain \( \pi_t \) from approaching 0. Low \( \chi \) implies a high probability of staying stagnant in the high regime and a low chance of jumping to the low regime, and low \( \lambda \) implies a high probability of becoming trapped in the low regime and a low chance of jumping to the high regime. When \( \lambda \) and(or) \( \chi \) become(s) greater, state beliefs will be more likely to display mean reversion and less stochastic variations. As a result, hedging motive declines and hedging demand decreases. Table 5 shows the simulation results.

Figure 5 plots the optimal portfolio allocation and the ambiguity-adjusted myopic stock allocation for an investor with different horizons from 1 to 120 months when \( \gamma \) is 2 and the probability of the high regime is set to its steady-state level. The left panel shows that for an expected utility (ambiguity-neutral) investor, the optimal allocation is strictly less than the myopic allocation, and the optimal allocation decreases with the horizon. This is the standard result in the model with learning, which has been found by Brennan (1998) and Honda (2003). Turning to the case under ambiguity, the middle panel shows that moderate degree of ambiguity tempers the horizon effect. That is, the optimal stock allocation still decreases with the horizon but decreases more slowly. However, from the right panel, we observe that very high degree of ambiguity can generate non-monotonic horizon dependence and increase the optimal stock allocation beyond the ambiguity-adjusted myopic allocation. This arises because the hedging demand for ambiguity dominates the hedging demand for estimation risk such that hedging demand becomes positive in sign. Moreover, an interesting result is that after certain length of the horizon, the optimal demand for stocks remains relatively constant regardless of the horizon.
4.3 Model Calibration

In this section, we calibrate the learning dynamics to the monthly market returns and the daily market returns. In particular, we consider an investor who learns about the unobservable Markov switching expected returns by observing the market returns over discrete time intervals. When applying the continuous-time filter to the discrete time data, we use the algorithm described in Appendix 6.2. The investor’s risk aversion parameter, $\gamma$, is assumed to be 8. The investment horizon is 10 years. This implies, for example, that the investor makes her portfolio and consumption decisions beginning in January 1997 with a targeted horizon date at December 2006, and faces a horizon date expanding over time. The steady state probabilities are used to set the initial prior that is to be updated through time. In the first calibration study, the monthly market returns span 120 months from January 1997 through December 2006. In the second study, the daily market returns are drawn from January 3rd 2006 to December 29th 2006 for all trading days in the year.

Figure 6 plots the optimal portfolio allocations and the consumption-wealth ratios for an expected utility investor and an ambiguity-averse investor, where the filtered probabilities are updated using the monthly returns. The ambiguity-averse investor prefers less aggressive portfolio strategies. When the market performs well and the undistorted state beliefs deliver significantly high expected returns, the expected utility investor is long in stocks while the ambiguity-averse investor takes “less long” positions. When the market is bad and it is profitable to sell short, the ambiguity-averse investor takes “less short” positions than the expected utility investor. In addition, when the market experiences several significant shocks, the consumption wealth ratios for the ambiguity-averse investor are higher than those for the expected utility investor. In this case, ambiguity improves state beliefs and the income effect dominates, leading to higher consumption-wealth ratios.

Figure 7 displays the results of the second calibration study, where the daily market returns are used to update beliefs. When $\kappa = 0$, the investor is ambiguity-neutral and hedging demand arises only due to estimation risk. The investor optimally holds less stocks than the myopic demand during the entire period considered here. When $\kappa > 0$, intertemporal hedging of ambiguity mitigates hedging demand for moderate degree of ambiguity aversion (for instance, $\kappa = 0.15$), making the optimal demand closer to the ambiguity-adjusted myopic demand. However, the sign of hedging demand can be reversed for relatively higher degree of ambiguity aversion (for example, $\kappa = 0.3$). In particular, when the investor believes that the likelihood of the market being in the high regime declines, she is
less sure about the current regime. As result, estimation risk is high and the effect of ambiguity on hedging demand becomes stronger. From Figure 7 (the panel of $\kappa = 0.3$), we notice that during the latter half of the period, ambiguity reverses the sign of hedging demand, making the optimal stock allocation beyond the myopic stock allocation. When ambiguity becomes even higher (for example, $\kappa = 0.5$), the optimal stock allocation stay above the myopic allocation for the entire period.

4.3.1 Welfare implication for learning

It has been shown in previous studies (Xia (2001) and Guidolin and Timmermann (2007)) that ignoring learning under incomplete information can incur a large welfare loss. In this section, we investigate the effect of ambiguity on the welfare loss from ignoring learning. Assume that the investor maximizes utility with respect to terminal wealth only. First, we compare the investor’s maximized expected utility accounting for learning to the utility of an investor who is constrained to choose either the i.i.d. strategy or the myopic strategy. The i.i.d. strategy is the portfolio strategy as if returns are i.i.d.. Following Xia (2001), we define the certainty equivalent wealth ($CEW$) of a portfolio strategy $s$ as the amount of wealth that makes the investor indifferent between receiving $CEW$ with certainty at horizon $T$ and having initial wealth $1$ to invest up to the horizon using strategy $s$:

$$e^{-\rho T} \frac{CEW_1 - \gamma}{1 - \gamma} = U(X_T \mid s) \quad X_0 = 1$$

where $U(X_T \mid s)$ is the expected utility under the reference probability measure $P$ when the strategy $s$ is adopted. Further, I define the present value of $CEW$ as $PVCEW_s = e^{-\gamma T} CEW_s$. The i.i.d. strategy is computed using the long run historical mean return and the sample variance. Figure 8 highlights the economic value of accounting for learning about regime switching mean returns. It shows that the optimal strategy, which correctly hedges for estimation risk, significantly increases the welfare, relative to the other two strategies. The present value of certainty equivalent wealth stays at approximately $5.5$ for the optimal strategy, while $3.3$ for the i.i.d. strategy and the myopic strategy.

Next, we define the certainty equivalent wealth ($\hat{CEW}$) of a particular portfolio strategy $\hat{s}$ under ambiguity as the amount of wealth that makes the investor indifferent between receiving $\hat{CEW}$ for sure at horizon $T$ and having initial wealth $1$ to invest up to the horizon using strategy $\hat{s}$ under
recursive multiple priors utility:

\[ e^{-\rho_T} \frac{\hat{C}E\hat{W}^{1-\gamma}}{1-\gamma} = V(X_T | \hat{s}) \quad X_0 = 1 \]

where \( V(X_T | \hat{s}) \) is the recursive multiple priors utility when the strategy \( \hat{s} \) is adopted. To examine the economic value of learning in the presence of ambiguity, the right panel in Figure 8 compares the \( P\hat{V}CEW \) for the optimal portfolio strategy, the i.i.d. strategy under ambiguity and the ambiguity-adjusted myopic strategy. Under ambiguity, the present value of certainty equivalent wealth stays at approximately $3 for the optimal strategy, while $2 for the i.i.d. strategy and the myopic strategy. Thus, the welfare loss from ignoring learning can be reduced by ambiguity.

5 Conclusion and Future Work

In this paper, we examine dynamic consumption and portfolio decisions under ambiguity in a continuous-time setting. The unobservable expected return follows a two-state continuous-time Markov chain. Recursive multiple priors utility is employed to accommodate the distinction between risk and ambiguity. By the martingale method and the Clark-Ocone formula, the optimal portfolio can be explicitly characterized by the Malliavin derivative and the expectations of stochastic integrals. We perform the Malliavin derivative based Monte Carlo (MDMC) simulations to compute the optimal portfolio and the consumption-wealth ratio. Learning introduces intertemporal hedging of ambiguity, which affects hedging demand. Under ambiguity, the optimal stock allocation decreases or varies non-monotonically with the investment horizon, depending on the degree of ambiguity aversion. Simulated examples show that ambiguity generates conservative and stable portfolio strategies. In addition, the welfare loss from ignoring learning can be reduced by ambiguity.

This study is among the few that examines dynamic consumption and portfolio choice taking into account both parameter uncertainty and model uncertainty. Although we have considered incomplete information and estimation risk, we exclude return predictability. Future research can combine uncertain predictability with model uncertainty. For instance, one can assume that the investor is ambiguous about the process for the predictive variable as well as the process governing the dynamics of the predictive relation. While in this paper, we use recursive multiple priors utility to capture ambiguity, future work can consider other recently axiomatized preferences that admit the distinction
between risk and ambiguity. For example, Klibanoff et al. (2006) axiomatize recursive smooth ambiguity preferences, and Maccheroni et al. (2006) axiomatize the preferences that are used in Hansen and Sargent’s robust control formulation.

6 Appendices

Theorem A1. (The Clark-Ocone formula)
Any random variable $F \in D^{1,2}$ can be decomposed as

$$F = E(F) + \int_t^T E [D_t F \mid \mathcal{F}_t] dB_t$$

where $\mathcal{F}_t$ represents the information filtration generated by the Brownian motion $B$ up to time $t$. A concise introduction to Malliavin calculus can be found in Oksendal (1997). Interested readers can refer to Nualart (1995) for the textbook treatment.

6.1 Proof of Theorem 3

Proof of Theorem 3[19] The first order conditions show that the optimal consumption and the terminal wealth can be characterized by the utility supergradients

$$e^{-\rho T} z^*_t (c^*_t)^{-\gamma} = y \xi_t$$

(6.1)

and

$$e^{-\rho T} z^*_T (X^*_T)^{-\gamma} = y \xi_T$$

(6.2)

from which we obtain the optimal consumption (2.15) and the optimal terminal wealth (2.16). Substituting (2.15) and (2.16) into the budget constraint gives the expression (2.17) for the Lagrange multiplier $y > 0$.

Applying Ito’s lemma to (2.19) gives us the dynamics of optimal consumption $c^*$

$$\frac{dc^*_t}{c^*_t} = \mu^*_t dt + \sigma^*_t d\hat{B}_t$$

(6.3)

[18] $D^{1,2}$ is the domain of the Malliavin derivative. See Nualart (1995) for further details.

[19] Another proof based on the generalized Clark-Ocone formula is available from the author on request.
where
\[ \mu^c_t = \frac{1}{\gamma}(r - \rho) + \frac{1}{2}(1 + \gamma)\sigma^2_c + \sigma_c \theta^*_t \] (6.4)
and
\[ \sigma^c_t = \frac{1}{\gamma}(\hat{\theta}_t - \theta^*_t) \] (6.5)

By homogeneity, we conjecture that along the optimal path, the utility process \( \{V_t(c^*, X_T)\} \) is given by
\[ V_t(c^*, X^*_T) = \left(\frac{c^*_t}{1 - \gamma}\right) A_t, \] (6.6)
with terminal condition
\[ V_T(c^*, X^*_T) = \left(\frac{X^*_T}{1 - \gamma}\right) A_T \]
where \( A_t \) satisfies the BSDE:
\[ \frac{dA_t}{A_t} = \mu^A_t dt + \sigma^A_t d\hat{B}_t, \quad A_T = 1. \]

An explicit formula for \( \sigma_A \) is given below.

Apply Ito’s Lemma to (6.6) delivers the following BSDE for the utility process
\[ dV_t = \mu^V_t dt + \sigma^V_t d\hat{B}_t \]

where
\[ \sigma^V_t = V_t \times [(1 - \gamma)\sigma^c_t + \sigma^A_t]. \] (6.7)

The functional form for \( \mu^V_t \) is irrelevant to the derivation below and thus omitted.

Now multiply both sides of the first order condition (6.1) by \( c^*_t \) and integrate over the product space \( dt \otimes dP \) to obtain
\[ E \left[ \int_0^T e^{-\rho t}(c^*_t)^{1-\gamma} \theta^*_t dt \right] = y E \left[ \int_0^T \xi_t c^*_t dt \right] \]

Also multiply both sides of the first order condition (6.2) by \( X^*_T \) and take expectation under probability measure \( P \) to produce
\[ E \left[ e^{-\rho T}(X^*_T)^{1-\gamma} \theta^*_T \right] = y E \left[ \xi_T X^*_T dt \right] \]
Then it follows

\[ X_0 = (1 - \gamma)(c_0^*)^\gamma V_0 \]

where we use the complementary slackness condition in the optimal control problem (2.14), the equality \( y = c_0^{-\gamma} \) and the definition of recursive multiple priors utility. In the same way, for all \( t \), we can deduce

\[ X_t^* = (1 - \gamma)(c_t^*)^\gamma V_t. \] (6.8)

Applying Ito’s Lemma to (6.8) and matching the volatility term with that in the budget constraint (2.7), we obtain an expression for the optimal portfolio \( \psi_t^* \)

\[ \psi_t^* = \frac{\sigma^c_t + \sigma^A_t}{\sigma} = \frac{1}{\gamma} \left( \hat{\mu}_t - r \right) - \frac{1}{\gamma} \frac{\theta_t^*}{\sigma} + \frac{\sigma^A_t}{\sigma} \] (6.9)

Next, we follow Ocone and Karatzas (1991) to derive an explicit formula for the optimal portfolio. The Martingale Representation Theorem implies that optimal wealth at time \( t \) satisfies

\[ \xi_t X_t^* = E_t \left[ \int_t^T \xi_\tau c_\tau^* d\tau + \xi_T X_T^* \mid \mathcal{F}_t^S \right]. \] (6.10)

By Ito’s lemma, the volatility of the left-hand side of (6.10) is \(-\xi_t X_t^* \hat{\theta}_t + \xi_t X_t^* \psi_t \sigma\). By the Clark-Ocone formula, the volatility of the right-hand side is given by \( E_t \left[ D_t \left( \int_t^T \xi_\tau c_\tau^* d\tau + \xi_T X_T^* \right) \right] \), where the conditional expectation is with respect to the filtration \( \mathcal{F}_t^S \). The two volatilities must be equal, which delivers

\[ \xi_t X_t^* \psi_t = \xi_t X_t^* \hat{\theta}_t \sigma^{-1} + E_t \left[ D_t \left( \int_t^T \xi_\tau c_\tau^* d\tau + \xi_T X_T^* \right) \right] \sigma^{-1} \] (6.11)

The Malliavin derivative of the right-hand side can be computed as follows. First, by linearity and exchangeability of the Malliavin derivative and the ordinary Lebesgue integral,

\[ D_t \left( \int_t^T \xi_\tau c_\tau^* d\tau + \xi_T X_T^* \right) = \int_t^T D_t (\xi_\tau c_\tau^*) d\tau + D_t (\xi_T X_T^*) \] (6.12)

We first consider the case of \( \theta_t^* = \kappa \). Then the second term of the right-hand side of (6.12) can be
computed as

\[ D_t (\xi T X^*_T) = X^*_T D_t \xi_T + \xi_T D_t X^*_T \]

in which

\[
D_t X^*_T = D_t \left( \left( e^{-\rho T \frac{x^*_T}{y T}} \right)^{\frac{1}{\gamma}} \right) = \frac{X^*_T}{\gamma} \left( \frac{\xi_T}{x^*_T} \right)^{-1} D_t \left( \frac{x^*_T}{\xi_T} \right)
\]

where the second and the third equality follow from the Chain rule of Malliavin calculus, and

\[
D_t z^*_T = D_t \exp \left( -\frac{1}{2} \int_0^T \kappa^2 ds - \int_0^T \kappa dB_s \right)
\]

where \( 1_{t \leq T} \) is the indicator function. Rearranging terms and assuming \( t \leq T \), we then have

\[ D_t (\xi T X^*_T) = -\frac{\kappa}{\gamma} \xi_T X^*_T + \frac{\gamma - 1}{\gamma} X^*_T D_t \xi_T \]

Similarly, suppose \( t \leq s \), we have

\[
D_t (\xi_s c^*_s) = -\frac{\kappa}{\gamma} \xi_s X^*_s + \frac{\gamma - 1}{\gamma} X^*_s D_t \xi_s
\]

Substituting (6.13) and (6.14) into (6.12), and rearranging terms and then applying the equality (6.11), we obtain

\[
\psi^*_t = \frac{\hat{\partial}_t}{\sigma} - \frac{\kappa}{\gamma \sigma} + \frac{\gamma - 1}{\gamma} \frac{1}{\sigma \xi_t X^*_t} E_t \left[ \int_t^T c^*_s D_t \xi_s dt + X^*_T D_t \xi_T \right]
\]

For \( t \leq s \), \( D_t \xi_s \) can be explicitly characterized as

\[
D_t \xi_s = -\xi_s \left( \hat{\partial}_t + \int_t^s \left( d\hat{B}_\tau + \hat{\partial}_\tau d\tau \right) D_t \hat{\partial}_\tau \right)
\]
Thus, the optimal portfolio \( \psi_t^* \) can be written as

\[
\psi_t^* = \frac{\hat{\theta}_t - \kappa}{\gamma \sigma} + \frac{1}{\gamma} \frac{1}{\sigma \xi_t X_t^*} E_t \left[ \int_t^T \xi_s c_s^* \left( \int_t^s D_t \hat{\vartheta}_\tau \left( d\tilde{B}_\tau + \hat{\vartheta}_\tau d\tau \right) \right) ds + \xi_T X_t^* \int_t^T \left( d\tilde{B}_s + \hat{\vartheta}_s ds \right) D_t \hat{\vartheta}_s \right]
\]

We can also rewrite the optimal portfolio under the probability measure \( \tilde{P} \), defined by Radon-Nikodym derivative \( d\tilde{P}/dP = \zeta_T \), that is,

\[
\psi_t^* = \frac{\hat{\mu}_t - r}{\gamma \sigma^2} - \frac{\kappa}{\gamma \sigma} + \frac{1}{\gamma} \frac{e^{rt}}{\sigma X_t^*} \tilde{E}_t \left[ \int_t^T e^{-rs} c_s^* \int_t^s \left( D_t \hat{\vartheta}_\tau d\tilde{B}_\tau \right) ds + e^{-rT} X_T^* \int_t^T D_t \hat{\vartheta}_s d\tilde{B}_s \right]
\]  \hspace{1cm} (6.15)

where the conditional expectation operator \( \tilde{E}_t \) is with respect to the probability measure \( \tilde{P} \), and \( \tilde{B} \) is defined by \( \tilde{B}_t = \hat{B}_t + \int_0^t \hat{\vartheta}_\tau d\tau \).

Finally, we give the condition such that \( \theta_t^* = \kappa \) and the optimal portfolio is given by (6.15). By (2.10), if \( \sigma_t^V = V_t \times ((1 - \gamma) \sigma_t^c + \sigma_t^A) > 0 \), then \( \theta_t^* = \kappa \). By (6.9) and (6.15), we have

\[
\sigma_t^A = \frac{1 - \gamma}{\gamma} \frac{e^{rt}}{X_t^*} \tilde{E}_t \left[ \int_t^T e^{-rs} c_s^* \int_t^s \left( D_t \hat{\vartheta}_\tau d\tilde{B}_\tau \right) ds + e^{-rT} X_T^* \int_t^T D_t \hat{\vartheta}_s d\tilde{B}_s \right]
\]

As a result, when

\[
(\hat{\theta}_t - \kappa) + \frac{e^{rt}}{X_t^*} \tilde{E}_t \left[ \int_t^T e^{-rs} c_s^* \int_t^s \left( D_t \hat{\vartheta}_\tau d\tilde{B}_\tau \right) ds + e^{-rT} X_T^* \int_t^T D_t \hat{\vartheta}_s d\tilde{B}_s \right] > 0
\]

\( \theta_t^* = \kappa \) holds.

In a similar way, we can show that when

\[
(\hat{\theta}_t + \kappa) + \frac{e^{rt}}{X_t^*} \tilde{E}_t \left[ \int_t^T e^{-rs} c_s^* \int_t^s \left( D_t \hat{\vartheta}_\tau d\tilde{B}_\tau \right) ds + e^{-rT} X_T^* \int_t^T D_t \hat{\vartheta}_s d\tilde{B}_s \right] < 0 ,
\]

\( \theta_t^* = -\kappa \) and the optimal portfolio is given by

\[
\psi_t^* = \frac{\hat{\mu}_t - r}{\gamma \sigma^2} + \frac{\kappa}{\gamma \sigma} + \frac{1}{\gamma} \frac{e^{rt}}{\sigma X_t^*} \tilde{E}_t \left[ \int_t^T e^{-rs} c_s^* \int_t^s \left( D_t \hat{\vartheta}_\tau d\tilde{B}_\tau \right) ds + e^{-rT} X_T^* \int_t^T D_t \hat{\vartheta}_s d\tilde{B}_s \right] .
\]

In the end, by applying the differentiation rules of Malliavin calculus to SDE (2.6), we can prove part (d). Thus far, we have completed the proof of Theorem 3.

Q.E.D.
6.2 Monte Carlo Simulation

When Monte Carlo simulation is implemented in solving financial problems, the time horizon $T$ is discretized into $n$ small intervals of size $\Delta t$, and $n$ pseudo-random numbers, $w_i, i = 1, \cdots, n$, are generated to simulate a possible path of a SDE. The pseudo-random numbers are drawn from a hypothetical normal distribution with zero mean and standard deviation $\sqrt{\Delta t}$ to approximate the instantaneous change of standard Brownian motion, $dB_t = B_t - B_{t-\Delta t}$. (For more on generating pseudo-random numbers, see Press et al. 1992). To simulate the optimal consumption and portfolio, we need to compute the expectation of stochastic integrals. In this setting, Monte Carlo simulation is an attractive method. For this model, Monte Carlo simulation can be implemented through the following steps:

(1) Euler Scheme: Discretize the evolution of relevant state variables by applying Euler’s scheme

$$z_{i+1}^\kappa = z_i^\kappa \exp\left(-\frac{1}{2} \kappa^2 \Delta t - \kappa w_i\right), \quad z_0^\kappa = 1$$

$$\zeta_{i+1} = \zeta_i \exp\left(-\frac{1}{2} \hat{\vartheta}_t^2 \Delta t - \hat{\vartheta}_t w_i\right), \quad \zeta_0 = 1$$

the perceived market price of risk $\hat{\vartheta}_t = (\hat{\mu}_t - r)/\sigma$, in which $\hat{\mu}_t$ evolves according to

$$\hat{\mu}_{i+1} = \lambda (\mu_H + \mu_L - 2\hat{\mu}_i) \Delta t + \frac{1}{\sigma} (\hat{\mu}_i - \mu_L)(\mu_H - \hat{\mu}_i) w_i$$

where $\hat{\mu}_0 = \mu_H \pi_0 + \mu_L (1 - \pi_0), \pi_0$ given. To discretize the evolution of Malliavin derivative $D_t \hat{\mu}_s$, let

$$H_{t,s} = \int_t^s D_t \hat{\vartheta}_\tau (dB_\tau + \hat{\vartheta}_\tau d\tau).$$

Applying Ito’s lemma and the chain rule of Malliavin derivatives gives

$$dH_{t,s} = (d\hat{B}_\tau + \hat{\vartheta}_s ds) \frac{1}{\sigma} D_t \hat{\mu}_s$$

where the evolution of $D_t \hat{\mu}_s$ can be discretized by Euler scheme. In the plain vanilla Monte Carlo simulation procedure, with the application of Euler discretization scheme, the precision of approximation relies on the number of Monte Carlo replications $M$ and the number of discretization points $N$. The size of the error is of the order $1/\sqrt{M} + 1/N$, and the convergence rate to the true values is
If we take $N$ proportional to $\sqrt{M}$ \[20\]

**(2) Variance stabilization:** Following Detemple et al. [2003], we adopt a reformulation of the discretized evolutions, which normalizes the stochastic terms to a constant in the simulated processes. This normalization also gives an ordinary differential equation (ODE) characterization for the term $D_t\hat{\mu}_t$. This transformation improves the convergence speed to the true values. In the current model setting, let $\mu^\hat{\mu}_t$ denote the drift term, and $\sigma^\hat{\mu}_t$ the volatility term of the process $\hat{\mu}_t$. We introduce a new state variable $Y_t = F(\hat{\mu}_t)$, where the function $F$ is chosen such that $\frac{dF}{d\hat{\mu}_t} = \frac{1}{\sigma^\hat{\mu}_t}$. Using Ito’s lemma shows that $Y$ satisfies the SDE

$$dY_t = m(Y_t)dt + dB_t$$

where $m(Y_t) = \frac{\mu^\hat{\mu}_t}{\sigma^\hat{\mu}_t} - \frac{1}{2} \frac{d\sigma^\hat{\mu}_t}{d\hat{\mu}_t}$ with $\hat{\mu}_t = F^{-1}(Y_t)$. The advantage of approximating this process instead of the original SDE for $\hat{\mu}_t$ is that we do not need to approximate the volatility term of the process. This improves the convergence properties of the Euler’s simulation scheme. Moreover, the Malliavin derivative $D_tY_t$ satisfies the ODE characterization

$$dD_tY_t = \frac{dm(Y_s)}{dY_s} D_tY_s ds, \quad D_tY_t = 1$$

with $D_t\hat{\mu}_s = \sigma^\hat{\mu}_s D_tY_s$. Thus the stochastic term is removed and simulating $D_t\hat{\mu}_s$ is reduced to the numerical approximation that has the same complexity as solving ODE. The one-to-one correspondence between $\hat{\mu}$ and $Y$ guarantees that $(\hat{\mu}_s, D_t\hat{\mu}_s)$ can be retrieved from $(Y_s, D_tY_s)$.

**(3)** We simulate $M$ sample path for all related state variables $(z^\hat{\mu}_t, \zeta_t, \mu_t, D_t\hat{\mu}_s, H_{t,s})$ starting from their initial values. Then to compute optimal consumption $(c_0^*)$ and portfolio$(\psi_0^*)$, we compute $M$ stochastic integrals along these $M$ sample paths according to the relevant state dynamics.

### 6.3 Discretization of Wonham Filter

This part gives a brief introduction to a discretization scheme for the Wonham filter, developed in Yin, Zhang and Liu (2004). The proof of convergence of the algorithm is also provided in Yin, Zhang and Liu (2004). This scheme is employed in the calibration exercise.

\[20\] See discussions in Detemple et al. (2003).
Consider the observable process $R$ satisfying the stochastic differential equation

$$dR(t) = \mu(t)dt + \sigma(t)dB(t)$$

where $\mu_t$ is a continuous-time finite state Markov chain with state space $N = \{\mu_1, \cdots, \mu_n\}$. $\mathcal{F}^R$ is the filtration generated by process $R$. The infinitesimal generator is given by $\Lambda = (\lambda_{ij}) \in \mathbb{R}^{n \times n}$. Then according to Lipster and Shiryayev (1977), the posterior probability

$$\pi_i(t) = \Pr(\mu(t) = \mu_i \mid \mathcal{F}^R(t))$$

satisfies the SDE

$$d\pi_i(t) = \sum_{j=1}^{n} \lambda_{ji} \pi_j(t)dt + \pi_i(t)(\mu_i - \bar{\mu})\sigma^{-1}(t)d\hat{B}(t)$$

with $\pi_i(0)$ given

where

$$\bar{\mu} = \sum_{i=1}^{n} \pi_i \mu_i$$

$$d\hat{B} = \frac{dR - \bar{\mu}dt}{\sigma}.$$ 

Let $v_i(t) = \log \pi_i(t)$, then the application of Ito’s lemma gives us

$$dv_i(t) = [\lambda_{ii} + \sum_{j=1}^{n} \lambda_{ji} \frac{\pi_j(t)}{\pi_i(t)}]dt - \frac{1}{2}\sigma^{-2}(t)(\mu_i - \bar{\mu}(t))^2dt + (\mu_i - \bar{\mu}(t))\sigma^{-1}(t)d\hat{B}(t)$$

(6.16)

If one applies the above filter to discrete data with constant step size $\Delta t$ and assumes the observations are generated according to

$$\Delta R(s) = R(s + 1) - R(s) = \mu(s)\Delta t + \sqrt{\Delta t}\sigma(s)\epsilon(s)$$

with one-step ahead transition matrix $P^{\Delta t} = I + \Delta t\Lambda$ where $\epsilon(s)$ is a sequence of white noise, then the discretized version of (7.13) yields the following algorithm

$$v_i(s + 1) = v_i(s) + \Delta t \times r_i(s) + \sqrt{\Delta t}\sigma^{-2}(s)(\mu_i - \bar{\mu}(s))\Delta R(s)$$
\[ v_i(0) = \log \pi_i(0), \quad \pi_i(0) \text{ given} \]

\[ r_i(s) = \lambda_{ii} + \sum_{j=1}^{n} \lambda_{ji} \frac{\pi_j(s)}{\pi_i(s)} - \sigma^{-2}(s)(\mu_i - \bar{\mu}(s))\bar{\mu}(s) - \frac{1}{2} \sigma^{-2}(s)(\mu_i - \bar{\mu}(s))^2 \]

\[ \pi_i(s + 1) = \frac{\exp(v_i(s + 1))}{\sum_{j=1}^{n} \exp(v_j(s + 1))} \]
References


### Table 1: Parameter estimates: a two-regime Markov switching model

<table>
<thead>
<tr>
<th>Parameter Descriptions</th>
<th>Notation</th>
<th>Parameter Values</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Discrete-time model</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High regime mean return</td>
<td>$m_H$</td>
<td>0.1872 (0.0189)</td>
</tr>
<tr>
<td>Low regime mean return</td>
<td>$m_L$</td>
<td>-0.0660 (0.0300)</td>
</tr>
<tr>
<td>Variance of market returns</td>
<td>$\sigma^2$</td>
<td>0.0083 (0.0021)</td>
</tr>
<tr>
<td>Transition probability from high-mean-return state to high-mean-return state</td>
<td>$P_{HH}$</td>
<td>0.6828 (0.1101)</td>
</tr>
<tr>
<td>Transition probability from low-mean-return state to low-mean-return state</td>
<td>$P_{LL}$</td>
<td>0.2628 (0.1216)</td>
</tr>
<tr>
<td><strong>Panel B: Continuous-time model</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High regime of mean return</td>
<td>$\mu_H$</td>
<td>0.1913</td>
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<tr>
<td>Low regime of mean return</td>
<td>$\mu_L$</td>
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<td>Volatility (standard deviation) of market returns</td>
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<td>High-mean-return density generator</td>
<td>$\chi$</td>
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<tr>
<td>Low-mean-return density generator</td>
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<td>Risk-free interest rate</td>
<td>$r$</td>
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<tr>
<td>Time preference parameter</td>
<td>$\rho$</td>
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</table>

This table shows the parameter estimates obtained by estimating the discrete-time analog of a two-state continuous-time Markov switching model.

\[ S_{t+1} = S_t \cdot e^{(\mu_H - \frac{1}{2} \sigma^2) \Delta t + \sigma \epsilon_{t+1}}, \quad \epsilon_{t+1} \sim N(0, 1) \]

where $\mu_H$ follows a two-state Markov Chain, and $m_H = \mu_H - \frac{1}{2} \sigma^2$. We set the time interval $\Delta t = 1$ and estimate the annual transition probability matrix, whose estimates and standard errors (in parentheses) are shown in Panel A. The estimation is done by applying an EM algorithm to compute the maximum likelihood estimates, as described in Hamilton (1989). Data are from CRSP U.S. historical monthly value-weighted market returns from January 1942 to December 2006. The real monthly market returns are compounded to obtain the annual market returns. The density generator matrix $\Lambda$ is estimated by

\[ \Lambda = \sum_{t=1}^{\infty} \frac{(-1)^{t+1}}{t} (P - I)^t \]

where the series is approximated using length 10.
Table 2: This table summarizes the effects of ambiguity on the optimal stock allocation, the myopic stock allocation, hedging demand and the consumption-wealth ratio for different risk aversions and priors of the high-regime mean return. The optimal stock allocation is defined by \( \psi_t^* = \frac{\hat{\mu}_t - r}{\gamma \sigma^2} + \frac{1 - \gamma}{\gamma} \frac{G_t}{\sigma X_t} \). The myopic allocation is defined by \( \psi_t^* = \frac{\hat{\mu}_t - r}{\gamma \sigma^2} \). Hedging demand is defined by \( 1 - \frac{\gamma}{\gamma} G_t \frac{1}{\sigma X_t} \). The investment horizon is 10 years.

All results are obtained by the Malliavin based Monte Carlo simulation method. Parameters used in the simulations are given in Table 1. We consider four values of the prior of the high-regime mean return: 0.1, 0.5, 0.7 and 0.9.

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Panel A: Optimal portfolio

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Panel B: Myopic portfolio

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Panel C: Hedging portfolio

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Table 3: This table summarizes the effects of ambiguity and the low-regime mean return on the optimal portfolio, the hedging portfolio and the consumption-wealth ratio. The investment horizon is 10 years. We consider three values of the prior of the high-regime mean return: 0.5, 0.7 and 0.9. For each value of $\mu_L$, we compute two sets of results for $\kappa = 0$ and $\kappa = 0.3$.

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Table 4: This table summarizes the effects of ambiguity and the return volatility on the optimal portfolio, the hedging portfolio and the consumption-wealth ratio. The investment horizon is 10 years. The wealth is $1. We consider three values of the prior of the high-regime mean return: 0.5, 0.7 and 0.9. For each value of $\sigma$, we compute two sets of results for $\kappa = 0$ and $\kappa = 0.2$.

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Table 5: This table summarizes the effects of ambiguity and transition densities on the hedging portfolio. The investment horizon is 10 years. The transition density parameter $\lambda$ ranges from 2.10 to 3.00, and the transition density parameter $\chi$ ranges from 0.60 to 1.50. Panel A presents the results for $\kappa = 0$. Panel B presents the results for $\kappa = 0.3$.

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<th>2.30</th>
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Figure 1: This figure plots hedging demand (top panel) and the optimal stock allocation (lower panel) for different risk aversions and ambiguity aversions. The horizon is 10 years and the prior of the high-regime mean return, $\pi_t$, is set to its steady state value. The risk aversion parameter, $\gamma$, ranges from 2 to 20. The ambiguity aversion parameter, $\kappa$, ranges from 0 to 0.6.
Figure 2: This figure plots hedging demand, the optimal stock allocation and the consumption-wealth ratio when $\pi_t$ takes values from 0.02 to 0.2. The risk aversion parameter, $\gamma$, is 4. The horizon is 10 years. Results are plotted for $\kappa = 0, 0.3$ and 0.4.

Figure 3: This figure plots the effects of $\pi_t$ and the ambiguity aversion parameter $\kappa$ on the two hedging demands (the hedging demand for estimation risk and the hedging demand for ambiguity), the optimal hedging demand, the optimal stock allocation, the ratio of hedging demand to the optimal demand for stocks and the consumption-wealth ratio. $\kappa = 0, 0.3$ and 0.4. The low-regime mean return is set to 0.0619. The horizon is 10 years and the risk aversion is 4.
Figure 4: This figure plots the effects of $\pi_t$ and the ambiguity aversion parameter $\kappa$ on the two hedging demands (the hedging demand for estimation risk and the hedging demand for ambiguity), the optimal hedging demand, the optimal stock allocation, the ratio of hedging demand to the optimal demand for stocks and the consumption-wealth ratio. $\kappa = 0, 0.3$ and 0.4. The low-regime mean return is set to 0.0619. The horizon is 10 years and the risk aversion is 8.

Figure 5: This figure plots the optimal stock allocations for investment horizons ranging from 1 month to 120 months. $\pi_t$ is set to its steady state value. The left panel plots the results for $\kappa = 0$. The middle panel plots the results for $\kappa = 0.4$. The middle panel plots the results for $\kappa = 0.8$. 
Figure 6: Simulated optimal stock allocations and consumption-wealth ratios using the historical monthly market returns: The upper panel plots the filtered state probabilities of the high-regime mean return. We apply the continuous-time Markov switching filter to the value-weighted monthly market returns from January 1997 to December 2006. The optimal stock allocations and the consumption-wealth ratios are plotted for $\kappa = 0$ and $\kappa = 0.12$ respectively in the middle panel and the lower panel. The risk aversion parameter, $\gamma$, is 8. The investment horizon is 10 years. The initial prior $\pi_0$ is set to its steady state value.
Figure 7: Simulated optimal stock allocations and consumption-wealth ratios using the historical daily market returns: The upper-leftmost panel plots the filtered state probabilities of the high-regime mean return. We apply the continuous-time Markov switching filter to the value-weighted monthly daily returns from January 3, 2006 to December 29, 2006. The optimal stock allocations and the myopic allocation are plotted for $\kappa = 0$, $\kappa = 0.3$ and $\kappa = 0.5$. The risk aversion parameter, $\gamma$, is 8. The investment horizon is 10 years. The initial prior $\pi_0$ is set to its steady state value.
Figure 8: Economic value of dynamic learning

The left panel plots the present value of certainty equivalent wealth (PVCEW) for the optimal portfolio strategy, the i.i.d. strategy and the myopic strategy for an expected utility investor. The right panel displays the present value of certainty equivalent wealth under ambiguity (\(\hat{P}VCEW\)) for the optimal portfolio strategy accounting for ambiguity, the i.i.d. strategy under ambiguity and the ambiguity-adjusted myopic strategy. Ambiguity aversion is 0.5. The risk aversion parameter, \(\gamma\), is 8. The investment horizon, \(T\), is 20 years. The initial prior \(\pi_0\) is set to its steady state value.