Disagreement about Inflation and the Yield Curve

Paul Ehling†  Michael Gallmeyer‡  Christian Heyerdahl-Larsen§  Philipp Illeditsch¶

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Abstract

We study how differences in beliefs about expected inflation impact real and nominal yield curves in a frictionless economy. Inflation disagreement induces a spillover effect to the real side of the economy with a strong impact on the real yield curve. When investors have a coefficient of relative risk aversion greater than one, real average yields across all maturities rise as disagreement increases. Real yield volatilities also rise with disagreement. Using the feature that nominal bonds can be computed from weighted-averages of quadratic Gaussian yield curves, increased inflation disagreement drives nominal yields and nominal yield volatilities higher at all maturities. Empirical support for these predictions is provided.

Keywords: Disagreement about expected inflation, feedback effect, real and nominal yields.

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†BI - Norwegian Business School, paul.ehling@bi.no.
‡The McIntire School of Commerce, University of Virginia, mgallmeyer@virginia.edu.
§London Business School, cheyerdahlarsen@london.edu.
¶The Wharton School, University of Pennsylvania, pille@wharton.upenn.edu.
1 Introduction

Several sophisticated reduced-form term structure models exist that are successful in explaining empirical features of U.S. Treasury bonds.\(^1\) However, the economic mechanisms driving these empirical regularities are not well understood.\(^2\) One logical candidate is how market participants view inflation. Indeed, since Friedman’s and Phelps’ work in the 1960s, inflation expectations have defined the core of monetary policy work and so naturally should impact bond prices. Mankiw, Reis, and Wolfers (2004) later argued that \textit{disagreement about inflation expectations} “may be a key to macroeconomic dynamics.” This is the departure point for our work where we study the role that disagreement about expected inflation plays in determining properties of real and nominal yield curves in a frictionless economy.

A key feature of our model is that disagreement about expected inflation impacts the equilibrium real pricing kernel. This spillover effect from the nominal to the real side of the economy is generated by the investors engaging in speculative trade in nominal bonds due to their differing opinions on inflation. Even if inflation is uncorrelated with economic fundamentals, the real pricing kernel is impacted by inflation disagreement — a feature like a sunspot equilibrium in Cass and Shell (1983) or Basak (2000).

We find that disagreement about expected inflation significantly impact the level of the real yield curve. When investors have a coefficient of relative risk aversion greater than one, the short rate increases in disagreement. Further, real average yields across all maturities rise as disagreement increases. When the coefficient of relative risk aversion is less than one, the real short rate and real average yields fall as disagreement increases. The direction of the shift in the real average yield curve is driven by the relative strength of income and substitution effects. Yield volatilities always increase with disagreement.

What is the intuition for the positive relation between yields and disagreement? Investors with

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\(^2\) Building from the reduced-form no arbitrage models, an intermediate approach has been to introduce macroeconomic variables into these no-arbitrage settings as in Ang and Piazzesi (2003), Ang, Dong, and Piazzesi (2007b), and Ang, Bekaert, and Wei (2008). Other works have explored full structural models. Wachter (2006), Piazzesi and Schneider (2007), and Bansal and Shaliastovich (2012) study structural term structure models with exogenous inflation. Basak and Yan (2010) incorporate money illusion in a setting with exogenous inflation. Buraschi and Jiltsov (2005), Gallmeyer, Hollifield, and Zin (2005), Gallmeyer, Hollifield, Palomino, and Zin (2007), Geanakoplos et al. (2009), Bekaert, Cho, and Moreno (2010), and Palomino (2010) study structural models that endogenize inflation. The recent surveys by Gürkaynak and Wright (2012) and Rudebusch (2010) summarize the implications of some of this structural work.
different beliefs about inflation trade nominal bonds to increase their consumption share in the future. When the coefficient of relative risk aversion is above one or the income effect dominates, then investors wish to increase their consumption today. Since consumption is exogenous it is not possible for all investors to increase consumption. Market clearing then implies that the short rate increases to counter balance consumption demands. A similar comparative static occurs in Epstein (1988) and Gallmeyer and Hollifield (2008). Importantly, we prove that this result on the short rate is propagated along the entire yield curve.

To quantify these effects for nominal yield curves, we derive closed-form term structures as weighted-averages of quadratic Gaussian term structure models. For this case, we numerically demonstrate that when disagreement about expected inflation increases, average nominal yields rise, the nominal yield curve flattens, and nominal yield volatilities increase. These impacts can be large. For a reasonable increase in differences in beliefs, average nominal yields at short maturities can rise by as much as 200 basis points.

We study the impact of inflation uncertainty on the equilibrium term structure of nominal and real interest rates and their volatilities in a pure exchange economy with external habit formation preferences. Our benchmark common beliefs model is adapted from the habit-formation settings of Abel (1990) and Chan and Kogan (2002). We employ habit-formation preferences to help match the level and slope of yields as well as the level of yield volatilities since inflation uncertainty cannot match these curves with plausible parameters.

We also empirically explore how differences in beliefs impact real and nominal term structure properties. We find support for increased inflation belief dispersion leading to higher real average yields and higher real yield volatilities. Further, we find that increased inflation belief dispersion leads to higher nominal yields and yield volatilities. Our empirical results remain after a series of robustness checks involving changes in specifications, sampling restrictions, and econometric methodologies. The yield volatility results are especially robust to using alternative data sources to proxy for inflation belief dispersion. Overall, our findings are consistent with the implications of our model of inflation belief dispersion.

As with any heterogeneous beliefs model, a common problem faced is linking investor beliefs to the true underlying economy. In fact, a common assumption is just to study the economy in a setting where one of the investors is assumed to have correct beliefs. Recent work by Piazzesi and Schneider
(2011) argues that difference in beliefs between investors and an econometrician might have additional implications for equilibrium quantities that depend on the reference probability measure. However, our results hold for all reference measures.

Our paper joins a growing literature that explores the role of subjective beliefs or survey data on the term structure. This work includes Ang, Bekaert, and Wei (2007a), Chernov and Mueller (2012), Chun (2011), and Piazzesi and Schneider (2012). Adrian and Wu (2010) in particular extract the term structure of inflation expectations by fitting an affine model of both real and nominal yield curves.

Through the speculative trade channel, disagreement about expected inflation impacts the equilibrium wealth distribution in our model. Other work directly appeals to how inflation can impact wealth distributions. Doepke and Schneider (2006) quantitatively explore the impact of inflation on the U.S. wealth distribution under two different assumptions about inflation expectations. Piazzesi and Schneider (2012), using an overlapping generations model with uninsurable nominal risk and disagreement about inflation, study the impact on wealth distributions due to structural shifts in the U.S. economy in the 1970s.

The closest works to ours are Xiong and Yan (2010) and Buraschi and Whelan (2010). Xiong and Yan (2010) build a model similar to ours. However, they employ logarithmic preferences for which income and substitution effects perfectly offset each other. Therefore, there is no spillover effect of inflation disagreement on real asset prices. The focus of their paper also differs from ours in that they study predictability while we consider other asset pricing properties such as the level and volatility of real and nominal yield curves. Buraschi and Whelan (2010) study how macroeconomic disagreement generates predictable variations in excess bond returns. They also use survey data on forecasts to test their predictions.

2 The Economy

To study the equilibrium impact of disagreement about inflation, our economic environment is a continuous-time pure exchange economy with heterogeneous investors. The economy has a finite horizon equal to $T$ with a single perishable consumption good. Real prices are measured in units of the consumption good and nominal prices are measured in dollars. Uncertainty is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$. 
The exogenous real aggregate output $\epsilon(t)$ process follows a geometric Brownian motion with dynamics given by

$$d\epsilon(t) = \epsilon(t) \left[ \mu_\epsilon \, dt + \sigma_\epsilon \, dz_\epsilon(t) \right], \quad \epsilon(0) > 0, \quad (2.1)$$

where $z_\epsilon(t)$ is a one-dimensional Brownian motion that represents a real shock. The exogenous price level $\pi(t)$ in the economy is given by

$$d\pi(t) = \pi(t) \left[ x(t) \, dt + \sigma_{\pi,\epsilon} \, dz_\epsilon(t) + \sigma_{\pi,\$} \, dz_\$(t) \right], \quad \pi(0) = 1, \quad (2.2)$$

where $x(t)$ denotes expected inflation and $z_\$(t)$ is a one-dimensional Brownian motion that represents a nominal shock to the economy. The Brownian motions $z_\epsilon(t)$ and $z_\$(t)$ are uncorrelated and $x(t)$ is unobservable. Since the price level also loads on $z_\epsilon(t)$, real aggregate output can be correlated with the price level.

We assume that $x(t)$ follows an Ornstein-Uhlenbeck process

$$dx(t) = \kappa (\bar{x} - x(t)) \, dt + \sigma_x \, dz_x(t), \quad x(0) \text{ given}, \quad (2.3)$$

where $x(0) \sim N(\bar{x}(0), \sigma_x^2(0))$, $dz_x(t) \frac{dx(t)}{\sigma_x(t)} = \rho_{x\epsilon} \, dt$, and $dz_x(t) \frac{dx(t)}{\sigma_x(t)} = \rho_{x\$(t)} \, dt$.

Nominal aggregate output in the economy is denoted by $\epsilon_\$(t) \equiv \pi(t)\epsilon(t)$ with dynamics given by

$$d\epsilon_\$(t) = \epsilon_\$(t) \left[ \mu_\$ \, dt + (\sigma_\$ + \sigma_{\pi,\$} \epsilon_\$) \, dz_\$(t) + \sigma_{\pi,\$} \, dz_\$(t) \right], \quad \epsilon_\$(0) > 0, \quad (2.4)$$

where $\mu_\$(t) \equiv \mu_\epsilon + x(t) + \sigma_\epsilon \sigma_{\pi,\epsilon}$.

### 2.1 Beliefs

Investors in the economy have heterogeneous beliefs about expected inflation. We assume that investors differ with respect to (i) the long run mean of expected inflation $\bar{x}$, (ii) the speed of mean reversion of expected inflation $\kappa$, or (iii) both.

For investors denoted as $i = \{1, 2\}$, uncertainty in the economy is represented by the filtered

\[\text{The analysis in Section 3 can be adopted to other dynamics for expected inflation such as finite-state Markov processes as in Veronesi (1999, 2000).}\]

\[\text{Heterogeneous beliefs are modeled with investor-specific priors about these quantities as in for example Detemple and Murthy (1994), Basak (2000), and Basak (2005).}\]
probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t^{x, \pi}\}, \mathcal{P}^i)\) where \(\mathcal{F}_t^{x, \pi}\) denotes the filtration generated by real output and the price level. Investors’ best estimate for expected inflation is

\[
x^i(t) = \mathbb{E}^i[x(t) \mid \mathcal{F}_t^{x, \pi}], \quad i \in \{1, 2\},
\]

where \(\mathbb{E}^i[\cdot]\) denotes the expectation with respect to investor \(i\)’s belief \(\mathcal{P}^i\).

From standard filtering theory as in Liptser and Shiryaev (1974a,b), investor \(i\)’s innovation processes for \(z_\epsilon(t)\) and \(z_\pi(t)\) are related to the reference probability \(\mathcal{P}\) via

\[
dz^i_\pi(t) = dz_\pi(t) + \frac{x(t) - x^i(t)}{\sigma_{\pi, \pi}} dt.
\]

Given the dynamics of real output \(\epsilon(t)\) are known, \(z_\epsilon(t)\) is known by all investors.

From equation (2.6), investors’ innovation processes are related to each other by

\[
dz^2_\pi(t) = dz^1_\pi(t) - \Delta(t) dt, \quad \Delta(t) = \frac{x^2(t) - x^1(t)}{\sigma_{\pi, \pi}}
\]

where the processes \(\Delta(t)\) represents disagreement across the two investors. This process summarizes investors’ differences in opinion about expected inflation. The disagreement is driven by initial priors and the paths of the realized real output and the price level.

Since no participant in the economy perfectly observes expected inflation, they infer it by observing the path of the price level \(\pi(t)\). From Liptser and Shiryaev (1974b), we can derive each market participant’s dynamics for the estimator of expected inflation \(x^i(t)\) where again we use \(i = 1, 2\) to denote the two investors. This filtering problem is summarized as follows.

**Proposition 1.** Under each investor’s beliefs, the price level is

\[
d\pi(t) = \pi(t) \left[ x^i(t) \, dt + \sigma_{\pi, \epsilon} \, dz_\epsilon(t) + \sigma_{\pi, \pi} \, dz^i_\pi(t) \right], \quad \pi(0) = 1,
\]

for \(i \in \{1, 2\}\).

Each investor’s beliefs about the expected inflation rate follows the process:

\[
dx^i(t) = \kappa^i \left( \bar{x}^i - x^i(t) \right) \, dt + \sigma^i_{x, \epsilon} \, dz_\epsilon(t) + \sigma^i_{x, \pi} \, dz^i_\pi(t), \quad x^i(0) \text{ given},
\]
for $i \in \{1, 2\}$ where $x^i(0) \sim N\left(\bar{x}^i(0), \sigma^2_{x^i}(0)\right)$. The volatility $\sigma_{x^i, \epsilon}$ is common across investors, $\sigma_{x^i, \epsilon} = \sigma_{x, \epsilon}$. The volatility $\sigma^i_{x, z}$ for $i = 1, 2$, given in the Appendix, differs across investors only when they disagree about the mean reversion of expected inflation, $\kappa^i$.

It follows from Proposition 1 and equation (2.7), that the dynamics of disagreement between the first and the second investor are

$$
\begin{align*}
d\Delta(t) &= \left(\frac{\kappa^2 \bar{x}^2 - \kappa^1 \bar{x}^1}{\sigma_{x, z}} + \frac{\sigma^2_{x, z} - \sigma^1_{x, z}}{\sigma_{x, z}^2} d^0(t) + \left[\frac{\kappa^1 - \kappa^2}{\sigma_{x, z}} + \frac{\sigma^1_{x, z} - \sigma^2_{x, z}}{\sigma_{x, z}^2}\right] x^1(t) - \left(\frac{\kappa^2 + \sigma^2_{x, z}}{\sigma_{x, z}}\right) \Delta(t)\right) dt + \frac{\sigma^2_{x, z} - \sigma^1_{x, z}}{\sigma_{x, z}} dz^0(t) \\
&= \left(\kappa^2 \bar{x}^2 - \kappa^1 \bar{x}^1\right) \sigma_{x, z}^{-1} dt + \left(x^1(t) - \phi(t)\right) dz^0(t) \quad \text{(2.10)}
\end{align*}
$$

The expected inflation rate’s instantaneous volatility $\sigma^i_{x, z}$ on the $z^i$ shock determines if $\Delta(t)$ is stochastic or deterministic. This process is deterministic when the investors only disagree about the long run mean of expected inflation $\bar{x}$. Disagreement about the mean reversion of expected inflation, $\kappa^i$, is the only channel that generates stochastic $\Delta(t)$ dynamics.

### 2.2 Security Markets

Investors trade continuously in a real riskfree asset, a nominal money market account, and a security whose real return is locally perfectly correlated with real consumption growth. For simplicity, we will refer to this security as a “stock.” This particular security structure is not crucial. We only require that the financial security structure be such that each investor can trade in a complete market.

Note that it is essential for our economic mechanism to work that investors can trade in nominal bonds. Without nominal bonds investors cannot implement their disagreement about inflation and the nominal shock is not priced.

The real risk-free asset is in zero-net supply and its real price is denoted by $B(t)$. The posited real price dynamics are

$$
\begin{align*}
dB(t) &= B(t) r(t) dt, \quad B(0) = 1, \\
&= \left(\kappa^2 \bar{x}^2 - \kappa^1 \bar{x}^1\right) \sigma_{x, z}^{-1} dt + \left(x^1(t) - \phi(t)\right) dz^0(t) \quad \text{(2.11)}
\end{align*}
$$

where $r(t)$ denotes the real riskfree rate to be determined in equilibrium. Investors observe the price of the real risk-free asset and hence know and agree on $r(t)$.

Let $S(t)$ denote the real price of a “stock” in zero net supply which is locally perfectly correlated
with real consumption growth and has a strictly positive volatility \( \sigma_{S, \epsilon}(t) > 0 \). The posited price dynamics are

\[
dS(t) = S(t) \left[ \mu_S(t) \, dt + \sigma_{S, \epsilon}(t) \, dz_{\epsilon}(t) \right], \quad S(0) = 1,
\]

where \( \mu_S(t) \) denotes the expected real rate of return on the stock under all investors’ beliefs given there is no disagreement about \( z_{\epsilon}(t) \).

The nominal money market account is in zero-net supply and its nominal price is denoted by \( P_S(t) \). The posited nominal price dynamics are

\[
dP_S(t) = P_S(t) \, r_S(t) \, dt, \quad P_S(0) = 1,
\]

where \( r_S(t) \) denotes the nominal risk-free rate to be determined in equilibrium. Both investors agree on the nominal price of the nominal money market account and hence know and agree on the nominal risk-free rate \( r_S(t) \).

Applying Itô’s lemma to the nominal price dynamics in equation (2.13) leads to the posited real price dynamics of the nominal money market account denoted by \( P(t) \equiv \frac{P_S(t)}{\pi(t)} \):

\[
\begin{align*}
dP(t) &= P(t) \left[ \mu_P(t) \, dt - \sigma_{\pi, \epsilon} \, dz_{\epsilon}(t) - \sigma_{\pi, S} \, dz_S(t) \right], \quad \mu_P(t) \equiv r_S(t) - x(t) + \sigma^2_{\pi, \epsilon} + \sigma^2_{\pi, S}, \\
dP(t) &= P(t) \left[ \mu^1_P(t) \, dt - \sigma_{\pi, \epsilon} \, dz_{\epsilon}(t) - \sigma_{\pi, S} \, dz^1_S(t) \right], \quad \mu^1_P(t) \equiv r_S(t) - x^1(t) + \sigma^2_{\pi, \epsilon} + \sigma^2_{\pi, S},
\end{align*}
\]

where equation (2.15) shows the real price dynamics of the nominal money market under each investor’s beliefs. Since both investors agree on the real price of the nominal money market \( P(t) \), the investor-specific expected returns are linked through

\[
\mu^1_P(t) - \mu^2_P(t) = \sigma_{\pi, S} \Delta(t) = x^2(t) - x^1(t).
\]

This difference in expected returns is solely driven by the disagreement about expected inflation.

The endogenous price system \((r(t), r_S(t), \mu_S(t))\) is determined in a dynamically complete equilibrium. It is convenient to summarize the price system in terms of investor-specific real state price

\footnote{By specifying this security to be non-dividend paying and in zero net supply, the volatility \( \sigma_{S, \epsilon}(t) \) can be taken as exogenous allowing for a dynamically complete price system.}
densities, or stochastic discount factors, that capture the investor-specific beliefs, but common Arrow-Debreu prices across investors. Investor $i$’s real state price density has dynamics

$$d\xi^i(t) = -\xi^i(t) \left[ r(t) \, dt + \theta^i(t) \, dz^\epsilon(t) + \theta^S_\pi(t) \, dz^\pi_\xi(t) \right], \quad \xi^i(0) = 1, \quad (2.17)$$

with

$$\theta^i(t) = \frac{\mu^S(t) - r(t)}{\sigma^S_\epsilon(t)}, \quad \theta^S_\pi(t) = -\frac{\mu^\pi_p(t) - r(t)}{\sigma^\pi_\xi(t)} - \frac{\sigma^\pi_\epsilon(t)}{\sigma^\pi_\xi(t)} \theta^\epsilon(t) \quad (2.18)$$

where $\theta^\pi_\xi(t)$ represents investor $i$’s perceived market prices of risk to the nominal shock.

Given investor 1 and 2 agree on the security prices as well as the real interest rate, the investor-specific market prices of risk of the nominal shock are linked through the disagreement process:

$$\theta^2_\pi(t) - \theta^1_\pi(t) = \Delta(t). \quad (2.19)$$

### 2.3 Investor Preferences and Consumption-Portfolio Choice Problem

Investors have “catching up with the Joneses” preferences as in Abel (1990) and Chan and Kogan (2002):

$$U^i = \mathbb{E}^i \left[ \int_0^T e^{-\rho t} \frac{1}{1-\gamma} \left( \frac{c^i(t)}{X(t)} \right)^{1-\gamma} \, dt \right], \quad i = \{1, 2\}, \quad (2.20)$$

where $\rho$ denotes the common subjective discount factor and $X(t)$ denotes the standard of living process. In equation (2.20), $\gamma$ measures the homogeneous local curvature of the utility function, i.e., the relative risk aversion parameter. We focus on a common CRRA-habit preferences assumption to isolate the effects of heterogeneous beliefs from heterogeneous risk aversion. Further, we use a habit-based preference structure to match moments of the yield curve, but quantitatively all results in Section 4 go through without habits.

The standard of living is measured as a weighted “geometric sum” of past realizations of aggregate output

$$\log(X(t)) = \log(X(0)) e^{-\delta t} + \delta \int_0^t e^{-\delta(t-a)} \log(e(a)) \, da, \quad \delta > 0, \quad (2.21)$$

where $\delta$ describes the dependence of $X(t)$ on the history of aggregate output.\(^6\) Defining relative log

\(^6\)If $\delta$ is large, then shocks to relative output are transitory and hence the standard of living process resembles closely
output as \( \omega(t) = \log(\epsilon(t)/X(t)) \), it follows a mean reverting process

\[
d\omega(t) = \delta(\bar{\omega} - \omega(t)) dt + \sigma \epsilon(t) dz(t), \quad \bar{\omega} = (\mu - \sigma^2/2)/\delta.
\] (2.22)

Investor \( i \) is endowed with a fraction of real aggregate output \( \epsilon^i > 0 \) where \( \epsilon^1(t) + \epsilon^2(t) = \epsilon(t) \).

The present value of investor’s wealth is given by

\[
W_i(0) = E_i \left[ \int_0^T \xi_i(t) \epsilon^i(t) dt \right].
\]

He then chooses a nonnegative consumption process \( c^i(t) \), and a portfolio process consisting of \( \psi^i_B(t) \) shares in the real risk-free asset, \( \psi^i_P(t) \) shares in the nominal money market account, and \( \psi^i_S(t) \) shares in the stock.

Complete markets allow the use of standard martingale techniques (Karatzas et al. (1987) and Cox and Huang (1989)) to solve the consumption-portfolio problem of each investor. The optimal consumption process \( \hat{c}^i(t) \) with supporting portfolio processes maximize the utility function given in equation (2.20) subject to the investor-specific static budget constraint

\[
E_i \left[ \int_0^T \xi^i(t) c^i(t) dt \right] \leq W^i(0).
\]

The optimal consumption process is

\[
\hat{c}^i(t) = X(t) \mathcal{I} \left( y^i e^{\sigma t} X(t) \xi^i(t) \right),
\]

where \( \mathcal{I}(\cdot) \) denotes the inverse function of \( \partial u(a)/\partial a \) and where the Lagrange multipliers \( y^i \) are determined from the investor-specific static budget constraints.

### 3 Equilibrium

Financial security prices and optimal allocations are characterized when investors disagree on the expected inflation rate and trade in nominal bonds.

**Definition 1.** Given preferences, endowments, and beliefs, an equilibrium is a collection of allocations\((c^1(t), \psi^1_B(t), \psi^1_S(t)), (c^2(t), \psi^2_B(t), \psi^2_S(t))\) and a price system \((r(t), \mu_S(t), r_S(t))\) such that \((\hat{c}^1(t), \psi^1_B(t), \psi^1_S(t))\) is an optimal solution to investor \( i \)'s consumption-portfolio problem given his perceived price processes, security prices are consistent across investors, and all markets clear for \( t \in [0, T] \). Specifically,

\[
c^1(t) + c^2(t) = \epsilon(t), \quad \psi^1_S(t) + \psi^2_S(t) = 0, \quad \psi^1_B(t) + \psi^2_B(t) = 0.
\]

The equilibrium can be constructed via a state-dependent representative agent as in Cuoco and He (1994) and Basak and Cuoco (1998) for example. The state-dependent representative agent at an current output; i.e. \( \omega(t) \approx 0 \). If \( \delta \approx 0 \), then shocks to relative output are persistent and hence past aggregate output receives high weight in the standard of living process.
arbitrary time $t$ is constructed by

$$U(\epsilon(t), X(t), \lambda(t)) = \max_{\{c^1(t) + c^2(t) = \epsilon(t)\}} \left( 1 + \gamma \left( \frac{c^1(t)}{X(t)} \right)^{1-\gamma} + \lambda(t) \frac{1}{1 - \gamma} \left( \frac{c^2(t)}{X(t)} \right)^{1-\gamma} \right).$$ (3.1)

**Proposition 2 (Equilibrium).** The equilibrium consumption allocations are

$$\hat{c}^1(\epsilon(t), \lambda(t)) = f(t) \epsilon(t), \quad \hat{c}^2(\epsilon(t), \lambda(t)) = (1 - f(t)) \epsilon(t),$$ (3.2)

where the consumption sharing rule is given by

$$f(t) = \frac{1}{1 + \lambda(t)^{\gamma}}.$$ (3.3)

The investor’s equilibrium state price densities are

$$\xi^1(\epsilon(t), X(t), \lambda(t)) = e^{-\rho t + \omega(t) - \omega(0)} \left( \frac{e^{\omega(t)} f(t)}{(e^{\omega(0)} f(0))} \right)^{-\gamma} \epsilon(t),$$ (3.4)

$$\xi^2(\epsilon(t), X(t), \lambda(t)) = \frac{1}{\lambda(t)} \xi^1(\epsilon(t), X(t), \lambda(t)) \lambda(0),$$ (3.5)

where the stochastic welfare weight $\lambda(t)$ has dynamics

$$d\lambda(t) = \lambda(t) \Delta(t) dz^1(t),$$ (3.6)

and $\lambda(0)$ solves either investor’s static budget constraint.

From the equilibrium construction of the real state price density for each investor, the equilibrium interest rate and market prices of risk can be computed highlighting the impact of belief heterogeneity about expected inflation on real prices.

**Proposition 3.** The dynamics of real and nominal state price densities, $\xi^1(t)$ and $\xi^1_S(t) = \frac{\xi^1(t)}{\pi(t)}$, of investor 1 are

$$d\xi^1(t) = -\xi^1(t) \left[ r(t) dt + \theta_\epsilon(t) dz_\epsilon(t) + \theta_\xi^1_S(t) dz^1_S(t) \right],$$ (3.7)

$$d\xi^1_S(t) = -\xi^1_S(t) \left[ r_S(t) dt + \theta_{\epsilon_S}(t) dz_\epsilon(t) + \theta_{\xi^1_S}(t) dz^1_S(t) \right],$$ (3.8)
where the real and nominal market prices of risk are as follows:

\[
\begin{align*}
\theta_\epsilon(t) &= \gamma \sigma_\epsilon, \\
\theta_\epsilon^1 (t) &= (f(t) - 1) \Delta(t), \\
\theta_\pi \epsilon (t) &= \gamma \sigma_\epsilon + \sigma_\pi \rho_\epsilon \pi, \\
\theta_\pi \epsilon^1 (t) &= \sigma_\pi \epsilon + (f(t) - 1) \Delta(t),
\end{align*}
\]

(3.9) \hspace{0.5cm} (3.10)

and the real and nominal interest rates are given by

\[
\begin{align*}
r(t) &= \rho + \mu_\epsilon - \frac{1}{2} (\gamma^2 + 1) \sigma_\epsilon^2 + \delta (\gamma - 1) (\bar{\omega} - \omega(t)) \\
&\quad + \frac{1}{2} \left(1 - \frac{1}{\gamma}\right) f(t) (1 - f(t)) (\Delta(t))^2, \\
r_\pi (t) &= r(t) + f(t) x_1(t) + (1 - f(t)) x_2(t) - \gamma \sigma_\epsilon \sigma_\pi \rho_\epsilon \pi - \sigma_\pi^2.
\end{align*}
\]

(3.11) \hspace{0.5cm} (3.12)

Proposition 3 highlights the impact of speculative trade on real prices in the economy. Although the investors in the economy do not disagree about any real quantities, disagreement about expected inflation, a nominal quantity, induces a spillover effect on the real side of the economy as the nominal shock \( z_\pi (t) \) is now priced through both the real interest rate \( r(t) \) and the market price of risk on the nominal shock \( \theta_i (t) \). In particular, once heterogenous beliefs are introduced, the real equilibrium interest rate \( r(t) \) is driven by the difference in beliefs between the two investors captured through \( \Delta(t) \) as long as the investors do not have logarithmic preferences as in Xiong and Yan (2010) for example. This mechanism, that heterogeneous beliefs about a nominal quantity can induce nominal risks to be priced on the real side of the economy, is distinct from New-Keynesian models such as Clarida, Gali, and Gertler (1999) where mechanisms such as sticky prices are imposed so that the nominal side of the economy impacts the real side of the economy.

When \( \gamma > 1 \) (\( \gamma < 1 \)), the equilibrium real interest rate is increasing (decreasing) with the difference in beliefs. Investors disagree about expected inflation and use nominal bonds to bet against each other. Both investors believe they will capture consumption from the other investor in the future. Classical income and substitution effects then impact the demand for consumption today as discussed in Epstein (1988) and Gallmeyer and Hollifield (2008) for example. Given consumption today is fixed, the real interest rate must adjust to clear markets. If \( \gamma = 1 \), the income and substitution effects exactly offset implying no impact on the real interest rate. When \( \gamma > 1 \), the real interest rate rises to counterbalance increased demand for borrowing. When \( \gamma < 1 \), the real interest rate falls to counterbalance lowered
demand for borrowing.

A real bond pays one unit of the consumption good at its maturity and a nominal bond pays one unit of currency at its maturity. Real bonds and nominal bonds are in zero net supply. All bonds considered in this paper are default-free zero-coupon bonds. Hence, the state price densities determined in Proposition 2 can be used to compute real and nominal bond prices.

Let $B(t; T')$ denote the real and $B_\$ (t; T') = B(t; T')\pi(t)$ the nominal price of a real (inflation-protected) bond maturing at $T'$. The real price of a real bond with maturity $T'$ is

$$B(t; T') = \mathbb{E}_t^i \left[ \frac{\xi_i(T')}{\xi_i(t)} \right]. \quad (3.13)$$

Let $P(t; T')$ denote the real and $P_\$ (t; T') = P(t; T)\pi(t)$ the nominal price of a nominal bond maturing at $T'$. The nominal price of a nominal bond with maturity $T'$ is

$$P_\$ (t; T') = \mathbb{E}_t^i \left[ \frac{\xi_i(T') \pi(t)}{\xi_i(t) \pi(T')} \right]. \quad (3.14)$$

Analogously, define the log-yields at time $t$ of a real and nominal zero-coupon bond with $\tau$ years to maturity as $y_B^{(\tau)} (t) = -\frac{1}{\tau} \log (B(t; t+\tau))$ and $y_{P_\$}^{(\tau)} (t) = -\frac{1}{\tau} \log (P_\$ (t; t+\tau))$, respectively. We provide closed-form solutions for real and nominal bond prices in the next section.

**Proposition 4.** Consider two economies at time $t$ where $\lambda(t)$ is identical across the two economies implying time $t$ consumption allocations are the same. Suppose one economy always exhibits more disagreement across the two investors than the other. Disagreement in the first economy, $\bar{\Delta}(s)$, then satisfies $|\bar{\Delta}(s)| \geq |\Delta(s)|$ for $s \geq t$ where $\Delta(s)$ denotes investor disagreement in the other economy.

Then, real bond prices $\bar{B}(t, T')$ and $\underline{B}(t, T')$ of maturity $T'$ in the higher and lower disagreement economies satisfy

$$\bar{B}(t, T') \begin{cases} > \underline{B}(t, T') & \text{if } \gamma < 1, \\ = \underline{B}(t, T') & \text{if } \gamma = 1, \\ < \underline{B}(t, T') & \text{if } \gamma > 1. \end{cases} \quad (3.15)$$

In particular, real bond yields increase with disagreement for $\gamma > 1$ for all $t$ and $T'$.

Proposition 4 highlights that the impact of increased disagreement on the real short rate propagates through the entire yield curve. When $\gamma > 1$, a case consistent with other observable equilibrium
price properties, the real yield curve increases as inflation disagreement increases in every state of nature.

**Proposition 5.** Consider the no disagreement benchmark economy. Suppose that time \( t \) consumption allocations are the same as in an economy with disagreement. Then the real bond volatility in the economy with disagreement rises as compared to the no disagreement benchmark.

4 Closed-Form Bond Prices

The previous results have highlighted qualitative properties of disagreement about expected inflation on equilibrium prices. To gain quantitative insights, it is useful to consider a setting where all bond prices can be computed in closed-form. To accomplish this, we assume that relative risk aversion \( \gamma \) is an integer. This assumption allows us to construct exact finite expansions of bond prices in artificial economies similar to equilibrium expansions computed in work such as Yan (2008), Dumas, Kurshev, and Uppal (2009), and Bhamra and Uppal (2010). Specifically, we decompose the real state price density through the following decomposition where economy \( k \) can be interpreted as a single investor habit formation economy with an aggregate endowment process given by \( \frac{\epsilon(t)}{\lambda(t)^{\gamma}} \).

**Proposition 6.** Assuming that \( \gamma \) is an integer, we can decompose the real state price density as

\[
\frac{\xi_{1}^{k}(t)}{\xi_{1}^{k}(0)} = \sum_{k=0}^{\gamma} w_{k}(0) \frac{\xi_{1}^{k}(t)}{\xi_{1}^{k}(0)},
\]

where \( \xi_{1}^{k}(t) \) can be interpreted as a real state price density in a fictitious economy given by

\[
\xi_{1}^{k}(t) = e^{-\rho t} \lambda(t)^{\gamma} \epsilon(t)^{1-\gamma} e^{(1-\gamma)\omega(t)}. \tag{4.2}
\]

The dynamics of \( \xi_{1}^{k}(t) \) are

\[
\frac{d\xi_{1}^{k}(t)}{\xi_{1}^{k}(t)} = -r_{k}(t)dt - \theta_{k,\epsilon}(t)d\epsilon(t) - \theta_{k,\omega}(t)d\omega(t), \tag{4.3}
\]
where

\[ r_k(t) = \rho + \gamma \mu_t - \frac{1}{2} \gamma (\gamma + 1) \sigma^2 - \delta (\gamma - 1) \omega(t) \]

\[ - \frac{1}{2} k \left( \frac{k}{\gamma} - 1 \right) \frac{1}{\sigma^2} (\Delta(t))^2, \]  

(4.4)

and

\[ \theta_{k,\epsilon}(t) = \gamma \sigma_t, \]  

(4.5)

\[ \theta_{k,\delta}(t) = \frac{1}{\sigma_{\pi,\delta}} \frac{k}{\gamma} \Delta(t). \]  

(4.6)

The quantity \( w_k(t) \) denotes the weight placed on \( \xi_k(t) \) and is driven by the sharing rule \( f(t) \):

\[ w_k(t) = \left( \frac{\gamma}{k} \right) \frac{\lambda(t)^k}{\left( 1 + \lambda(t)^{\frac{1}{\gamma}} \right)^{\gamma}} = \left( \frac{\gamma}{k} \right) f(t)^{\gamma-k} (1 - f(t))^k \]  

(4.7)

with \( \sum_{k=0}^{\gamma} w_k(t) = 1. \)

Likewise, we can also decompose the nominal state price density when \( \gamma \) is an integer as

\[ \frac{\xi_{k\delta}^1(t)}{\xi_{k\delta}^1(0)} = \sum_{k=0}^{\gamma} w_k(0) \frac{\xi_{k\delta}^1(t)}{\xi_{k\delta}^1(0)}, \]  

(4.8)

where \( \xi_{k\delta}^1(t) = \xi_{k\delta}(t) / \pi(t) \) and \( \xi_{k\delta}^1(t) = \xi_{k\delta}(t) / \pi(t) \). The dynamics of \( \xi_{k\delta}^1(t) \) are summarized in the following corollary.

**Corollary 1.** The nominal stochastic discount factor in artificial economy \( k \) is defined as \( \xi_{k\delta}^1(t) = \xi_{k\delta}^1(t) / \pi(t) \). We then have

\[ \frac{d\xi_{k\delta}^1(t)}{\xi_{k\delta}^1(t)} = - r_{k\delta}(t) dt - \theta_{k\delta,\epsilon} dz_{\epsilon}(t) - \theta_{k\delta,\delta}^1 dz_{\delta}^1(t), \]  

(4.9)

where

\[ r_{k\delta}(t) = r_k(t) + \left( 1 - \frac{k}{\gamma} \right) x^1(t) + \frac{k}{\gamma} x^2(t) - \gamma \rho_{\epsilon \pi} \sigma_\pi \sigma_\epsilon - \sigma^2, \]  

(4.10)

\[ \theta_{k\delta,\epsilon}(t) = \gamma \sigma_\epsilon + \rho_{\epsilon \pi} \sigma_\pi, \]  

(4.11)
and
\[ \theta^{1}_{k,\$}(t) = \frac{1}{\sigma_{\pi,\$}} k \Delta(t) + \sigma_{\pi,\$}. \] (4.12)

These state price decompositions allow us to interpret each fictitious economy \( k \) as a single investor economy where the difference in beliefs is captured through a fictitious aggregate endowment process. The weighting of each fictitious state price density to recover the actual state price density is solely driven by the sharing rule \( f(t) \). By decoupling the sharing rule in the artificial economy, we can express the real and nominal bond price as a finite sum of real and nominal bond prices in artificial economies.

The real price of a real zero-coupon bond is therefore
\[ B(t; T') = \sum_{k=0}^{\gamma} w_k(t) B_k(t; T'), \] (4.13)
where \( B_k(t; T') \) denotes the real price of a real bond in artificial economy \( k \) given by
\[ B_k(t; T') = E_t \left[ \frac{\xi_1^{1}(T')}{\xi_1^{k}(t)} \right] = E_t \left[ \frac{\xi_1^{1}(T')}{\xi_1^{k}(t)} \right] | \omega(t) = \omega, x^1(t) = x^1, x^2(t) = x^2]. \] (4.14)

Likewise, the nominal price of a nominal zero-coupon bond is therefore
\[ P_{\$}(t; T') = \sum_{k=0}^{\gamma} w_k(t) P_{\$k}(t; T'), \] (4.15)
where \( P_{\$k}(t; T') \) denotes the nominal price of a nominal bond in artificial economy \( k \) given by
\[ P_{\$k}(t; T') = E_t \left[ \frac{\xi_{1\$}^{1}(T')}{\xi_{1\$}^{k}(t)} \right] = E_t \left[ \frac{\xi_{1\$}^{1}(T')}{\xi_{1\$}^{k}(t)} \right] | \omega(t) = \omega, x^1(t) = x^1, x^2(t) = x^2]. \] (4.16)

Given the structure of the artificial economies, we now show that the artificial real and nominal term structures are in the class of quadratic Gaussian term structure models as studied in Ahn, Dittmar, and Gallant (2002). To show this mapping, we adopt largely the same notation as Ahn, Dittmar, and Gallant (2002) for the state vector \( Y(t) \) in the economy, where \( Y(t) = (x^1(t), x^2(t), \omega(t))' \). Additional details are given in the Appendix.

The real bond prices in artificial economy \( k \) follow a quadratic Gaussian term structure model and are summarized in the following proposition.
Proposition 7. The real and nominal bond prices, \( B_k(t; T') \) and \( P_{k\$,}(t; T') \), in the artificial economy \( k \) are an exponential quadratic functions of the state vector given by

\[
B_k(t; T') = \exp \left\{ A_k(T' - t) + B_k(\tau) Y(T'-t) + Y(t) C_k(T' - t) Y(t) \right\},
\]
\[
P_{k\$,}(t; T') = \exp \left\{ A_{k\$,}(T' - t) + B_{k\$,}(\tau) Y(T'-t) + Y(t) C_{k\$,}(T' - t) Y(t) \right\},
\]

where the coefficients are the solutions to ordinary differential equations summarized in the Appendix.

Summarizing, when both habit-utility investors are endowed with an integer risk aversion \( \gamma \), real and nominal bond prices can be expressed as expansions of artificial economics with quadratic Gaussian term structures. The weights in the expansions are driven by the sharing rule \( f(t) \) providing an additional channel to impact bond prices and their dynamics.

4.1 Quantitative Impact of Disagreement about Inflation on Yields

We now explore how properties of the yield curve are quantitatively impacted by differences of beliefs in a numerical example. We assume that \( \bar{x} = 3\% \) and \( \kappa = 0.3 \). Table 1 outlines the other parameters used for the example. Investors have a risk aversion coefficient of \( \gamma = 7 \). They have different beliefs about the dynamics of inflation both through the long run mean \( \bar{x} \) and the speed of mean reversion \( \kappa \). We assume that the investors are dogmatic about the long run mean \( \bar{x}^i \) and the speed of mean reversion \( \kappa^i \). Specifically, \( \bar{x}^1 = 2.5\% \) and \( \bar{x}^2 = 3.5\% \) and hence the difference in beliefs about the long run mean is \( \Delta_{\bar{x}} = 0.01 \) and the speed of mean reversion is \( \Delta_{\kappa} = -0.3 \). Both investors learn from real output and the price level about expected inflation and thus their steady state estimates are 2.79 for investor 1 and 3.09 for investor 2, respectively; this corresponds to a steady state disagreement of 30bp.

Figure 1 plots the average nominal yields and nominal yield volatilities as a function of maturity in the data and as predicted by the model.\(^7\) In these plots, the sharing rule is set such that \( f(t) = 0.5 \) with belief dynamics computed in the steady state where \( x^1(t) = 0.0279 \) and \( x^2(t) = 0.0309 \). In addition to plotting the data and the difference in beliefs economy (HE-Model), the two figures also plot two representative investor economies under either investor 1’s or investor 2’s beliefs. From the left plot in Figure 1 with the introduction of the habit we learn that the average yield curve is now

\(^7\)Based on several numerical trials, the numerical results presented appear insensitive to the signs of the disagreement over \( \bar{x} \) and \( \kappa \).
upward sloping and roughly consistent with the data. The yield volatilities for the data and the models are given in the right plot in Figure 1. For the models, we plot the instantaneous volatilities, while the data plotted is for monthly frequency yield volatilities. Due to time aggregation, it is not surprising that the instantaneous volatilities are below the monthly volatilities from the data. Across the models, the difference in beliefs economy always shows the highest yield volatilities relative to the representative investor models.

With this base case of the yield curve established, we now ask how heterogeneous beliefs impacts both the average yield curve and yield volatilities in Figure 2. In addition to the steady state case, we consider two other cases — high disagreement and no disagreement. In both these cases, the two investors still disagree about the underlying model driving expected inflation through a different $\bar{x}$ and a different $\kappa$. In the high disagreement case, the spread between the two beliefs increases — $x^1(t) = 0.0253$ and $x^2(t) = 0.0336$. In the no disagreement case, the belief spread for expected inflation for the two investors collapses to zero — $x^1(t) = 0.03$ and $x^2(t) = 0.03$.

From these plots, the amount of disagreement has a strong impact on both the nominal and the real yield curve. When disagreement goes down, yields go down, yield curves steepen, and yield volatilities fall. When disagreement goes up, nominal yields rise and the slopes of the yield curves flatten. In this particular example, the yield curves even invert. Additionally, yield volatilities increase as disagreement increases. While disagreement increases bond yield volatility, it also flattens the average yield curve.

Quantitatively, the example also highlights that the impact of disagreement about inflation on the real and nominal yield curve can be large. For example, the nominal yield curve at a maturity of one year shifts by over 200 basis points when moving between the no disagreement and the high disagreement case. At longer maturities, the impact lessens. At a five year maturity, the shift is slightly more than 50 basis points. If expected inflation is very persistent ($\kappa \approx 0$), then an increase in disagreement has the same effect on short term and long term yields. However, if expected inflation shows little persistence or $\kappa$ is large, then disagreement has almost no effect on long term yields.

Figure 3 shows the impact of the sharing rule on the average real and nominal yield curves respectively under the state disagreement. The real curve highlights the impact of the spillover effect.

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8Without the habit, the average yield curve is downward sloping. See for example Backus and Zin (1994).
9Based on our numerical work, the feature that increased disagreement increases nominal average yields and volatilities as well flattens the yield curve seems robust across a large parameter space.
Given the two investors disagree about the expected inflation dynamics, their speculative trade spills over to the real side of the economy. In the plot, we see that the yield curve slopes are maximized when the consumption sharing between the two investors is roughly equal.

### 4.2 Indirect Effect of Inflation Expectations on Real Yields

We now study the effects of inflation expectations on real bond yields for a given disagreement. The four graphs in Figure 4 show real bond yields as a function of expected inflation as perceived by the first investor. All graphs show the real short rate and real bond yields with maturities ranging from one to five years assuming that the disagreement between the two investors is fixed at its long run mean.

Expected inflation has no effect on the real short rate, but it still affects real bond yields as long as there is disagreement about $\kappa$. If there is no disagreement about $\kappa$ (top left graph of Figure 4), then the disagreement process is deterministic and expected inflation does not predict future disagreement and hence does not predict future consumption shares. However, if there is disagreement about $\kappa$, then expected inflation predicts future disagreement and therefore affects real yield levels. For instance, the top right graph of Figure 4 shows real yields, if there is no disagreement about the long run mean. In this case both investors agree on the current expected inflation rate and the long run mean of 3%. However, they disagree on the speed of mean reversion and therefore current inflation expectations that deviate from its long run mean predict high future disagreement. This leads to a U-shaped relation between expected inflation and real yields with minimum at the common long run mean of 3%. If investors disagree about the long run mean and the speed of mean reversion (bottom row of Figure 4), then the relation is still U-shaped but the minim shifts upwards (downwards) if the disagreement about the long run mean and the speed of mean reversion has the same (opposite) sign.

### 5 Empirical Evidence

Given the model’s linkage between inflation disagreement and yield curve properties, we empirically explore these relationships using U.S. Treasury bond data. We construct our sample of nominal yields from the CRSP Risk-Free Rates File (1- and 3-month yields based on average bid/ask prices) and the Fama-Bliss Discount Bond File (yields with 1 to 5 years to maturity based on artificial discount
bonds). Both data sets have a long time span and are widely used. To complement this data with yields with longer maturities, we use data from Gürkaynak, Sack, and Wright (2007) (yields with 6 to 30 years to maturity) fitted using the Svensson (1994) extension of the Nelson and Siegel (1987) methodology. The real yield sample merges the implied real rates in Chernov and Mueller (2012) (Q3 1971 to Q4 2002) with the TIPS data in Gürkaynak et al. (2010). From these yield series, we estimate a GARCH(1,1) for yield volatilities. We structure our yield and yield volatility data series at monthly, quarterly and semi-annual frequencies.

To build a measure of disagreement from expected inflation forecasts, we use three commonly used surveys of economic data — the Livingston Survey, the Michigan Surveys of Consumers, and the Survey of Professional Forecasters. From the raw series of forecasts, we compute the mean forecast (“Mean Inflation”) as well as the standard deviation around the mean forecast, which we call “Dispersion,” at each point in time. Dispersion is a proxy for inflation disagreement in our model. The Michigan Surveys of Consumers data is monthly (available since January 1978), the Survey of Professional Forecasters data is quarterly (available since September 1981), and the Livingston Survey data is semianual (available since December 1946). Below we focus on results obtained from employing the Michigan Surveys of Consumers data.

Using realized inflation and Mean Inflation as proxy for expected inflation, we also construct our own proxies for real yields by subtracting it from nominal yields as discussed below. At the short end of the yield curve, this approximation of real yields is expected to work well given that the variation in the inflation risk premium should be small at short horizons. Again, from these real yield series, we estimate a GARCH(1,1) for yield volatilities.

Figure 5 shows the term structure of real and nominal yields and yield volatilities for the periods 1978 to 2011 and April 1981 to 2011. The second period April 1981 to 2011 excludes the high inflation years. Mean real and nominal yields increase with maturity, but the term structures flatten out for yields between three and five year maturities (and even more so for longer maturities). In contrast, average real and nominal yield volatilities decrease with maturity. These features of the yield and yield volatility curves are standard and appear not to depend on the high inflation period.

Figure 6 shows the evolution of mean beliefs and dispersion for the Michigan Surveys of Consumers.

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11For some maturities, observations are missing between the implied real data in Chernov and Mueller (2012) and the TIPS data in Gürkaynak, Sack, and Wright (2010). Estimating a GARCH(1,1) for yield volatilities on two separate series or on a merged series that ignores missing observations does not affect our regression results.
over time with NBER recessions as gray shaded areas. In the top panel, the monthly one year ahead CPI is plotted in addition to the mean survey forecast. We see from the figure that the mean of the survey forecast appears to contain valuable information regarding future realizations of inflation. In other words, the mean inflation forecast predicts realized inflation as in Ang, Bekaert, and Wei (2007a). Yet, we also see from the figure that consumers at times are surprised by low realizations of inflation. Nevertheless, we do not find that professional forecasters outperform consumers. Specifically, in monthly predictive regressions, we find the following Newey-West corrected t-statistics on Mean Inflation 2.85, 12.81, and 2.78 for the Livingston Survey, the Michigan Surveys of Consumers, and the Survey of Professional Forecasters, respectively. The adjusted $R^2$ of these regressions are 19.3, 28.3, and 3.0. Ang et al. (2007a) use RMSE to discriminate on predictive ability. Following their approach, we find that the mean forecast of each survey shows predictive power for inflation and that performance is similar across surveys. These predictive regressions show similar relative performance at quarterly, semi-annual, and annual frequencies and appear insensitive to alternative specifications including replacing the mean forecast with the median.

From the bottom panel in Figure 6, we note a rather high inflation belief dispersion derived from the Michigan Surveys of Consumers. The high dispersion embedded in the data of the Michigan Surveys of Consumers, also relative to other surveys such as the Livingston Survey and the Survey of Professional Forecasters, need not necessarily be surprising, considering that the Michigan Survey asks questions about price changes from the perspective of households. Since households have arguably different consumption bundles, this probably implies increased dispersion relative to a hypothetical survey that asks questions about the CPI instead. In addition, Malmendier and Nagel (2011) argue that dispersion in consumer forecast data is higher than in the professional forecaster data as older consumers consistently overestimate both inflation and its volatility based on past experience. Figure 6 supports this view as the path of the mean inflation forecast almost always lies above realized inflation after the high inflation period at the beginning of our sample.

Figure 7 shows the mean of nominal and real yields curves that are sorted on belief dispersion. We compute curves for five buckets each containing 20% of the dispersion distribution. Consistent with our model the yield curve from the top 20% dispersion bucket shows the highest yields for all maturities. The next two buckets with lower dispersion also produce yield curves that line up as predicted by our model. However, the order of the yield curves for the two bottom buckets is reversed.
both for nominal and real yields. These sorts are robust to excluding observations from the beginning of our sample to remove any influence from high inflation.

Next, Figure 8 shows the mean of nominal and real yield volatility curves that are sorted on belief dispersion. Again, we compute curves for five buckets. The resulting dispersion-sorted volatility curves appear also to be consistent with our model. Similar to the nominal mean yield curves, the two lowest dispersion buckets lead to an ordering of the nominal volatility curves that is reversed for parts of the curves. For real yield volatilities, all curves line up as predicted by our model.

Overall, sorted yield and yield volatility curves provide support for our model predictions. We note that almost all of the differences in means, data point by data point and curve by curve, are highly statistically significant (with p-value’s of 0.00) and are always jointly significant with a p-value of 0.00. All differences in mean yields and mean yield volatilities are also significant with a p-value of 0.00 between the top 20% and the bottom 20% dispersion portfolios. Test results hold for nominal and real yields and are not affected by excluding observations from high inflation period. The p-values of the tests are available from the authors.

Our main prediction is that inflation belief dispersion drives up real yields and yield volatilities. Testing these predictions further through a regression analysis should be fruitful especially because (high) inflation (risk) cannot mechanically drive up real yields and yield volatilities. Unfortunately, the TIPS time series is too short to be of great use for our purpose.\textsuperscript{12} Instead, we employ the quarterly data from Chernov and Mueller (2012) together with the TIPS data to put our prediction to the test.\textsuperscript{13} Panel A and B of Table 2 present coefficient estimates from these regressions for the following maturities: 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, and 7-year. For each maturity, the table presents two models. Model 1 contains a constant and Dispersion as explanatory variables. Model 2 includes Mean Inflation as additional explanatory variable to control for feedback effects as in Section 4.1. We see that in Panel A the sign of the coefficient, for Model 1 regressions, is always positive consistent with our prediction. The Newey-West corrected t-statistics are significant for 2-year and 3-year maturities. Overall, the t-statistics are high for maturities beyond 1-year, probably because of the increase in the number of observations. Model 2 shows similar pattern with significant

\textsuperscript{12}We only have 96 monthly observations using the TIPS data. Yield regressions show the correct sign, but all coefficients are insignificant. Yield volatility regressions always show the correct sign and have highly statistically significant coefficient estimates up to the 20-year maturity.

\textsuperscript{13}In this specification, we use TIPS data from 2004 on for 2-year and 3-year maturities and interpolate with cubic splines the rates for 2003. For 5-year and 7-year maturities, we use TIPS data from 2003 onwards.
Dispersion coefficients starting from 2-year maturity. Panel B of Table 2 presents the estimates of the
dispersion regressions for yield volatilities for Models 1 and 2. We see that all coefficient estimates
for Dispersion are significant and positive.

To address concerns regarding the small number of observations, we supplement the above results
with imputed real yields computed from nominal yields and realized inflation over the inflation forecast
horizon. Panel A and B of Table 3 presents coefficient estimates from the two regression models for
the following maturities: 1-month, 3-month, 1-year, 2-year, 3-year, 4-year, and 5-year. We see that
in Panel A the sign of the coefficient is always positive consistent with our prediction. For Model 1
the Newey-West corrected t-statistics are significant at the short end of the yield curve, until 2-year,
consistent with our assertion that imputed real yields should work well as long as the variation in the
inflation risk premium is small. The Newey-West corrected t-statistics are significant for all maturities
for Model 2. Additionally, all regressions with yield volatilities as the dependent variable, in Panel
B, show the expected sign with highly significant coefficient estimates.

Our next check is to ask whether disagreement about expected inflation between investors, as
measured by inflation forecast dispersion, shows a significantly positive relation with nominal bond
yields in regression models. Panel A of Table 4 presents estimates of these dispersion regression
coefficients for the following maturities: 1-month, 3-month, 1-year, 2-year, 3-year, 4-year, and 5-year.
For each maturity, the table presents two models. Model 1 contains a constant and Dispersion as
explanatory variables. We recognize that periods with high inflation imply high dispersion. At least
for nominal yields, such a mechanical relation can lead to a rejection of the null hypothesis simply
because high dispersion implies high inflation which obviously implies higher nominal bond yields.
Model 2 addresses this concern by including Mean Inflation as an explanatory variable. Mean Inflation
is also a proxy, albeit a rough one, for the consumption-weighted mean inflation forecast, which is a
state variable in our model. All coefficients of Dispersion in Panel A of Table 4 have positive signs and
are highly statistically significant as the smallest Newey-West corrected t-statistic is as high as 3.976.
We then ask if Dispersion shows a significantly positive relation with nominal bond yield volatilities.
Panel B of Table 4 presents estimates of these dispersion regression coefficients. Again, the table
presents Model 1 and Model 2 for each maturity. As for yields, all coefficients of Dispersion in Panel
B of Table 4 have positive signs and are highly statistically significant.

Several robustness checks, including tests using the Livingston Survey and the Survey of Profes-
sional Forecasters, are available from the authors.\textsuperscript{14} Two robustness checks related to our model and one check related to the persistence of the series are worth mentioning however. First, the model contains one more state variable, namely relative log output $\omega$. Since $\omega$ is uncorrelated with dispersion and the consumption-weighted inflation forecast in our theoretical model, both regression models should remain well-specified regressions. Nevertheless, when we include the log price/dividend ratio of the US stock market as proxy for $\omega$, we find that the regression results in Table 3 and Table 4 are unchanged and that the log price/dividend ratio shows the predicted negative sign. Second, one theoretical concern with our regressions is if dispersion is correlated with inflation volatility, which might be — contrary to our model — priced. However, our regression results are robust to the inclusion of inflation volatility estimated through a GARCH(1,1). Third, all our time series show high persistence, which casts doubts about the Newey-West corrected t-statistics. We therefore include lagged Dispersion in a series of robustness checks to soak up the persistence in the data. Yet, the sign of the coefficient estimates on Dispersion and the significance of the Newey-West corrected t-statistics in these regressions remain the same. From our main regressions and robustness tests, we find that the empirical results are largely statistically significant, robust, and consistent with the model’s theoretical predictions.

6 Conclusion

We study how differences in beliefs about expected inflation affect the real and nominal term structures. Our model shows that differences in beliefs about expected inflation impact the equilibrium real pricing kernel generating a spill-over effect from the nominal to the real side of the economy. We find that heterogeneous beliefs about expected inflation have a strong impact on the level and volatility of the real yield curve. When both investors share common preferences over consumption with a coefficient of relative risk aversion greater than one, real average yields across all maturities rise as disagreement increases. When the coefficient of relative risk aversion is less than one, real average yields fall as disagreement increases. Over both cases, yield volatilities increase with dis-

\textsuperscript{14}The yield regressions using the Livingston Survey and the Survey of Professional Forecasters show the correct signs for Dispersion but result frequently in insignificant coefficient estimates, possibly due to the low frequency of the data and due to too few observations. Yield volatility regressions, however, always show positive and significant coefficient estimates for Dispersion. We also obtain a copy of the mean inflation forecast and the inflation dispersion based on the Blue Chip Financial Forecasts data from Chun (2011). Again, yield volatility regressions, consistently produce positive and highly significant coefficient estimates for Dispersion.
agreement. For additional nominal term structure implications, we consider a simplifying case where the term structures can be computed in closed-form as a consumption-weighted quadratic Gaussian term structure model. We demonstrate numerically how the nature of the difference in beliefs about inflation among investors is important in generating features of the real and nominal yield curves. From empirical work, we find support for our model’s predictions about the relationship between inflation disagreement and yield curve properties for both real and nominal term structures.
Appendix

Proofs and Auxiliary Results

Proof of Proposition 1. The result follows from Theorem 12.1 of Liptser and Shiryaev (1974b). The volatilities $\sigma_{x,\epsilon}^i$ and $\sigma_{x,\pi}^i$ for $i = 0, 1, 2$ are

$$\sigma_{x,\epsilon}^i = \sigma_{x,\epsilon} = \sigma_x \rho_{x\epsilon}, \quad (6.1)$$

$$\sigma_{x,\pi}^i = \frac{\sigma_x}{\sqrt{1 - \rho_{x\pi}^2}} \left( \rho_{x\pi} - \rho_{x\pi}\rho_{x\epsilon} + \frac{1}{\sigma_x \sigma_{\pi}} v^i \right), \quad (6.2)$$

where $v^i$ is agent $i$’s estimation error.

Suppose the estimation error $v^i$ is equal to its steady state value, i.e., it is a constant, given by

$$a (v^i)^2 + b^i v^i + c = 0, \quad (6.3)$$

with

$$a = -\frac{1}{(1 - \rho_{x\pi}^2) \sigma_{\pi}^2}, \quad (6.4)$$

$$b^i = -2 \kappa_i^i - \frac{2 \sigma_x}{\sigma_{\pi} (1 - \rho_{x\pi}^2)} (\rho_{x\pi} - \rho_{x\pi}\rho_{x\epsilon}), \quad (6.5)$$

$$c = \frac{\sigma_x^2}{1 - \rho_{x\pi}^2} (1 - \rho_{x\pi}^2 - \rho_{x\epsilon}^2) + 2 \rho_{x\pi} \rho_{x\epsilon} \rho_{x\epsilon}). \quad (6.6)$$

Proof of Proposition 2. The proof follows from Karatzas et al. (1990) with the appropriate modifications taken to accommodate for investors facing different state prices through heterogeneous beliefs.

Proof of Propositions 3. The proof follows from applying Itô’s lemma to each investor’s first order conditions, imposing market clearing, and match coefficients in the dynamics of the real and nominal state price densities.

Proof of Proposition 4. The real bond price written in terms of investor 1’s beliefs is given by

$$B(t; T') = E_t^1 \left[ \frac{\xi^i(T')}{\xi^i(t)} \right]$$

$$= e^{-\rho(T-t)} E_t^1 \left[ \left( 1 + \frac{\lambda(T)^{\frac{1}{2}}} {1 + \lambda(t)^{\frac{1}{2}}} \right)^{-\gamma} \left( \frac{\epsilon(T)}{\epsilon(t) \left( 1 + \lambda(T)^{\frac{1}{2}} \right)^{\gamma}} \right)^{-\gamma} e^{(1-\gamma)(\omega(T)-\omega(t))} \right].$$

Given we only focus on differences in beliefs about inflation, $\lambda(t)$ and $\epsilon(t)$ are uncorrelated implying

$$B(t; T') = e^{-\rho(T-t)} E_t^1 \left[ \left( \frac{\epsilon(T)}{\epsilon(t)} \right)^{-\gamma} e^{(1-\gamma)(\omega(T)-\omega(t))} \right] \times E_t^1 \left[ \left( 1 + \frac{\lambda(T)^{\frac{1}{2}}} {1 + \lambda(t)^{\frac{1}{2}}} \right)^{\gamma} \right].$$
Increasing disagreement only impacts the last expectation. First, note that real bond prices are more volatile under disagreement as under the benchmark of no disagreement, \( \lambda(t) \) is a constant.

To establish how increased disagreement impacts the expectation given by

\[
E^1_t \left[ \frac{(1 + \lambda(T) \gamma)}{(1 + \lambda(t) \gamma)} \right],
\]

we can apply the comparison theorem stated below due to Hajek (1985) where the weighting process \( \lambda(t) \) is a martingale.

**Theorem 1** (Mean Comparison Theorem Adapted from Hajek (1985)). Let \( x \) be a continuous martingale with representation

\[
x(t) = x(0) + \int_0^t \sigma(s) dw(s)
\]

such that for some Lipschitz continuous function \( \rho \), \( |\sigma(s)| \leq \rho(x(s)) \) and let \( y \) be the unique solution to the stochastic differential equation

\[
y(t) = x(0) + \int_0^t \rho(y(s)) dw(s).
\]

Then, for any convex function \( \Phi \) and any \( t \geq 0 \),

\[
E[\Phi(x(t))] \leq E[\Phi(y(t))].
\]

**Proof of Proposition 6.** The proof follows by applying Proposition 3.

**Proof of Proposition 7.** The proof directly follows from applying Proposition 8 in the Appendix. Mapping into the Ahn et al. (2002) setting, the dynamics of \( Y(t) = (x^0(t), x^1(t), x^2(t), \omega(t))^T \) are

\[
dY(t) = (\mu + \xi Y(t)) dt + \Sigma dZ_2(t),
\]

where

\[
\mu = (\kappa_0 \bar{x}_0, \kappa_1 \bar{x}_1, \kappa_2 \bar{x}_2, \delta \bar{\omega})^T \in \mathbb{R}^4,
\]

\[
\xi = \begin{pmatrix}
-\kappa_0 \\
0 \\
\sigma_{x,s}^1 \\
0
\end{pmatrix}
- \begin{pmatrix}
\kappa_1 + \frac{\sigma_{x,s}^1}{\sigma_{x,s}} \\
0 \\
\sigma_{x,s} \\
0
\end{pmatrix}
\begin{pmatrix}
\kappa_2 + \frac{\sigma_{x,s}^2}{\sigma_{x,s}} \\
0
\end{pmatrix}
\in \mathbb{R}^{4 \times 4},
\]

\[
\Sigma = \begin{pmatrix}
\sigma_{x,\epsilon}^0 & \sigma_{x,s}^0 \\
\sigma_{x,\epsilon}^1 & \sigma_{x,s}^1 \\
\sigma_{x,\epsilon}^2 & \sigma_{x,s}^2 \\
\sigma_{\epsilon} & 0
\end{pmatrix}
\in \mathbb{R}^{4 \times 2},
\]

and

\[
Z_2(t) = (z_{\epsilon,0}(t), z_{s,0}(t))^T \in \mathbb{R}^2.
\]

The volatilities \( \sigma_{x,\epsilon}^i \) and \( \sigma_{x,s}^i \) for \( i = 0, 1, 2 \) are given in Proposition 1.
The coefficients for the real bond price, $A_k(T' - t)$, $B_k(\tau)$, and $C_k(T' - t)$, are the solutions of the ordinary differential equations given in Proposition 8 where

$$
\eta_{0,k} = -(\gamma \sigma_e, 0)' \\
\eta_{Y1,k} = 0_4 \\
\eta_{Y2,k} = -\frac{1}{\sigma_{\pi,\$}} \left( 1, -\left(1 - \frac{k}{\gamma}\right), -\frac{k}{\gamma}, 0 \right)' \\
\alpha_k = \rho + \gamma \mu_e - \frac{1}{2} \gamma (\gamma + 1) \sigma_e^2 \\
\beta_k = (0, 0, 0, \delta(1 - \gamma))' \\
\Psi_k = -\frac{1}{2} \left( \frac{k}{\gamma} - 1 \right) \frac{1}{\sigma_{\pi,\$}^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

The matrix $\Psi_k$ is positive semidefinite because $k/\gamma \leq 1$. Note $\psi_k$ is singular and $\psi_{k\$} = 0_{4\times4}$ if $k = 0$ or $k = \gamma$.

The coefficients for the nominal bond price, $A_k\$(T' - t)$, $B_k\$(\tau)$, and $C_k\$(T' - t)$, are the solutions of the ordinary differential equations given in Proposition 8 where

$$
\eta_{0,k\$} = -(\gamma \sigma_e + \rho \epsilon \sigma_{\pi,\$}, \sigma_{\pi,\$})' \\
\eta_{Y1,k\$} = 0_4 \\
\eta_{Y2,k\$} = -\frac{1}{\sigma_{\pi,\$}} \left( 1, -\left(1 - \frac{k}{\gamma}\right), -\frac{k}{\gamma}, 0 \right)' \\
\alpha_{k\$} = \rho + \gamma \mu_e - \frac{1}{2} \gamma (\gamma + 1) \sigma_e^2 - \gamma \rho \epsilon \sigma_e \sigma_{\pi} - \sigma_{\pi}^2 \\
\beta_{k\$} = \left( 0, 1 - \frac{k}{\gamma}, \frac{k}{\gamma}, \delta(1 - \gamma) \right)' \\
\Psi_{k\$} = -\frac{1}{2} \left( \frac{k}{\gamma} - 1 \right) \frac{1}{\sigma_{\pi,\$}^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

The matrix $\Psi_{k\$}$ is positive semidefinite because $k/\gamma \leq 1$. Note that $\psi_{k\$}$ is singular and $\psi_{k\$} = 0_{4\times4}$ if $k = 0$ or $k = \gamma$.

\[\square\]

**Quadratic Gaussian Term Structure Models**

Here we use the same notation as Ahn, Dittmar, and Gallant (2002).\(^{15}\) Let $Y(t)$ denote a $N-$dimensional vector of state variables and $Z_M(t)$ a $M-$dimensional vector of independent Brownian motions.

\(^{15}\)In contrast to Ahn, Dittmar, and Gallant (2002): (i) we assume that the vector of Brownian motions driving the discount factor is identical to the vector of Brownian motions driving the state variables and thus $\Upsilon$ is the identify matrix, and (ii) we allow the vector of Brownian motions to have a dimension that is different from the number of state variables.
Assumption 1. The dynamics of the stochastic discount factor $SDF(t)$ are\(^{16}\)

$$\frac{dSDF(t)}{SDF(t)} = -r(t) \, dt + 1'_M \, \text{diag} \left[ \eta_0m + \eta'_YmY(t) \right]_M \, dZ_M(t) \quad (6.24)$$

with

$$\eta_0 = (\eta_{01}, \ldots, \eta_{0M})' \in \mathbb{R}^M \quad (6.25)$$

$$\eta_Y = (\eta_{Y1}, \ldots, \eta_{YM})' \in \mathbb{R}^{M \times N} \quad (6.26)$$

Hence, the market price of risk is an affine function of the state vector $Y(t)$.

Assumption 2. The short rate is a quadratic function of the state variables:

$$r(t) = \alpha + \beta'Y(t) + Y(t)'\Psi Y(t), \quad (6.27)$$

where $\alpha$ is a constant, $\beta$ is an $N$-dimensional vector of constants, and $\Psi$ is an $N \times N$ dimensional positive semidefinite matrix of constants.\(^{17}\)

If the matrix $\Psi$ is non singular, then $r(t) \geq \alpha - \frac{1}{2} \beta'\Psi^{-1}\beta \forall t$.

Assumption 3. The state vector $Y(t)$ follows a multidimensional OU-process:

$$dY(t) = (\mu + \xi Y(t)) \, dt + \Sigma dZ_M(t), \quad (6.28)$$

where $\mu$ is an $N$-dimensional vector of constants, $\xi$ is an $N$-dimensional square matrix of constants, and $\Sigma$ is an $N \times M$-dimensional matrix of constants. We assume that $\xi$ is diagonalizable and has negative real components of eigenvalues. Specifically, $\xi = U\Lambda U^{-1}$ in which $U$ is the matrix of $N$ eigenvectors and $\Lambda$ is the diagonal matrix of eigenvalues.

Let $V(t, \tau)$ denote the price of a zero-coupon bond and $y(t, \tau)$ the corresponding yield. Specifically,

$$V(t, \tau) = E_t \left[ \frac{SDF(t+\tau)}{SDF(t)} \right] \quad (6.29)$$

$$y(t, \tau) = -\frac{1}{\tau} \ln (V(t, \tau)). \quad (6.30)$$

The bond price and corresponding yield are given in the next proposition.

Proposition 8 (Quadratic Gaussian Term Structure Model). Let $\delta_0 = -\Sigma \eta_0 = -\Sigma \eta_0$ and $\delta_Y = -\Sigma \eta_Y = -\Sigma \eta_Y$. The bond price is an exponential quadratic function of the state vector

$$V(t, \tau) = \exp \left\{ A(\tau) + B(\tau)'Y(t) + Y(t)'C(\tau)Y(t) \right\}, \quad (6.31)$$

\(^{16}\)An apostrophe denotes the transpose of a vector or matrix, $1'_M$ denotes a vector of ones, and $\text{diag} [Y'_m]_M$ denotes an $M$-dimensional matrix with diagonal elements $(Y_1, \ldots, Y_m)$.

\(^{17}\)We don’t impose an additional parameter restriction that guarantees non-negativity of the short rate.
where $A(\tau), B(\tau),$ and $C(\tau)$ satisfy the ordinary differential equations,

$$
\frac{dC(\tau)}{d\tau} = 2C(\tau)\Sigma\Sigma' C(\tau) + (C(\tau)(\xi - \delta_Y) + (\xi - \delta_Y)' C(\tau)) - \Psi
$$

(6.32)

$$
\frac{dB(\tau)}{d\tau} = 2C(\tau)\Sigma\Sigma' B(\tau) + (\xi - \delta_Y)' B(\tau) + 2C(\tau)(\mu - \delta_0) - \beta
$$

(6.33)

$$
\frac{dA(\tau)}{d\tau} = \text{trace}[\Sigma\Sigma'C(\tau)] + \frac{1}{2}B(\tau)'\Sigma\Sigma'B(\tau) + B(\tau)'(\mu - \delta_0) - \alpha,
$$

(6.34)

in which $A(0) = 0, B(0) = 0_N, \text{ and } C(0) = 0_{N \times N}. \text{ Moreover, the yield is a quadratic function of the state vector } Y(\tau):$

$$
y(t, \tau) = A_y(\tau) + B_y(\tau)' Y(t) + Y(t)' C_y(\tau) Y(t)
$$

(6.35)

with $A_y(\tau) = -A(\tau)/\tau, B_y(\tau) = -B(\tau)/\tau, \text{ and } C_y(\tau) = -C(\tau)/\tau.$

**Proof.** See Ahn et al. (2002). □

If the short rate is an affine function of the state vector $Y(\tau),$ then the bond price is an exponential affine function of the state vector $Y(\tau)$ because $\Psi = 0_{N \times N}$ implies $C(\tau) = 0_{N \times N}$ for all $\tau.$ The bond price in this case belongs to the class of essential affine term structure models (see Duffee (2002)) and is given in the next corollary.

**Proposition 9** (Essential Affine Term Structure Model). Let $\Psi = 0_{N \times N}, \delta_0 = -\Sigma \eta_0 = -\Sigma \eta_0$ and $\delta_Y = -\Sigma Y \eta_Y = -\Sigma Y \eta_Y$ and assume that $(\xi - \delta_Y)$ is invertible. The bond price is an exponential affine function of the state vector

$$
V(t, \tau) = \exp\{A(\tau) + B(\tau)' Y(t)\},
$$

(6.36)

where

$$
B(\tau) = -((\xi - \delta_Y))^{-1}(e^{(\xi - \delta_Y)'} - I_{N \times N}) \beta,
$$

(6.37)

$I_{N \times N}$ denotes the $N$ dimensional identity matrix, and

$$
A(\tau) = \frac{1}{2} \beta' \left( \int_0^T (e^{(\xi - \delta_Y)'} u) K e^{(\xi - \delta_Y)'} du \right) \beta
$$

$$
- \beta' K - (\mu - \delta_0)' ((\xi - \delta_Y))^{-1} \left( \int_0^T e^{(\xi - \delta_Y)'} u du \right) \beta
$$

$$
+ \left( \frac{1}{2} \beta' K \beta + (\mu - \delta_0)' ((\xi - \delta_Y))^{-1} \beta - \alpha \right) \tau
$$

(6.38)

with

$$
K = \left( ((\xi - \delta_Y))^{-1} \right)' \Sigma \Sigma' ((\xi - \delta_Y))^{-1}.
$$

(6.39)

If $(\xi - \delta_Y)$ is diagonalizable; i.e. $(\xi - \delta_Y) = T \Lambda T^{-1}$ then\(^{18}\)

$$
B(\tau) = -T \text{diag} \left[ \frac{1}{\lambda_i} \left( e^{\lambda_i \tau} - 1 \right) \right] T^{-1} \beta,
$$

(6.40)

$$
\int_0^T e^{(\xi - \delta_Y) u} du = T \text{diag} \left[ \frac{1}{\lambda_i} \left( e^{\lambda_i \tau} - 1 \right) \right] T^{-1},
$$

(6.41)

\(^{18}\)The matrix $(\xi - \delta_Y)$ is invertible and thus all eigenvalues are nonzero.
and

$$\int_0^\tau \left(e^{(\xi-\delta Y)u}\right)' K e^{(\xi-\delta Y)u} \, du = (T^{-1})' G(\Lambda, t) T^{-1}, \quad (6.42)$$

where $G(\Lambda, t)$ is a $m \times m$-matrix with elements given by

$$G_{ij} = \frac{\omega_{ij}}{\lambda_i + \lambda_j} \left( e^{(\lambda_i + \lambda_j)t} - 1 \right) \quad (6.43)$$

and $\omega_{ij}$ denotes the element of the matrix $\Omega = T'KT$ in the $i$th-row and $j$th-column.

Moreover, the yield is an affine function of the state vector $Y(t)$:

$$y(t, \tau) = A_y(\tau) + B_y(\tau)' Y(t) \quad (6.44)$$

with $A_y(\tau) = -A(\tau)/\tau$, and $B_y(\tau) = -B(\tau)/\tau$.

Proof. where $A(\tau)$ and $B(\tau)$ satisfy the ordinary differential equations,

$$\frac{dB(\tau)}{d\tau} = (\xi - \delta Y)' B(\tau) - \beta \quad (6.45)$$

$$\frac{dA(\tau)}{d\tau} = \frac{1}{2} B(\tau)' \Sigma \Sigma' B(\tau) + B(\tau)' (\mu - \delta_0) - \alpha, \quad (6.46)$$

in which $A(0) = 0$ and $B(0) = 0_N$. \qed
Table 1: **Parameter Choice for Two Habit Investor Example.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>Time Preference Parameter</td>
<td>2.5%</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Risk Aversion</td>
<td>7</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Habit Parameter</td>
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<tr>
<td>$\mu_\epsilon$</td>
<td>Expected Consumption Growth</td>
<td>1.72%</td>
</tr>
<tr>
<td>$\sigma_\epsilon$</td>
<td>Volatility of Consumption</td>
<td>3.32%</td>
</tr>
<tr>
<td>$\sigma_\pi$</td>
<td>Inflation Volatility</td>
<td>1.3%</td>
</tr>
<tr>
<td>$\bar{x}$</td>
<td>Long Run Mean of Expected Inflation</td>
<td>3%</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Mean Reversion of Expected Inflation</td>
<td>0.35</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td>Volatility of Expected Inflation</td>
<td>1.4%</td>
</tr>
<tr>
<td>$\rho_{\epsilon \pi}$</td>
<td>$\rho$ of Realized Inflation &amp; Real Consumption Growth</td>
<td>−0.2</td>
</tr>
<tr>
<td>$\rho_{\pi x}$</td>
<td>$\rho$ of Realized and Expected Inflation</td>
<td>0</td>
</tr>
<tr>
<td>$\rho_{x \epsilon}$</td>
<td>$\rho$ of Expected Inflation &amp; Real Consumption Growth</td>
<td>0</td>
</tr>
<tr>
<td>$\bar{x}_1$</td>
<td>Long run mean of first investor</td>
<td>$\bar{x} - \frac{1}{2} \Delta \bar{x}$</td>
</tr>
<tr>
<td>$\bar{x}_2$</td>
<td>Long run mean of second investor</td>
<td>$\bar{x} + \frac{1}{2} \Delta \bar{x}$</td>
</tr>
<tr>
<td>$\kappa_1$</td>
<td>Mean reversion of first investor</td>
<td>$\kappa - \frac{1}{2} \Delta \kappa$</td>
</tr>
<tr>
<td>$\kappa_2$</td>
<td>Mean reversion of first investor</td>
<td>$\kappa + \frac{1}{2} \Delta \kappa$</td>
</tr>
</tbody>
</table>
Table 2: Inflation Beliefs Dispersion and Real Yields I. The table reports results from OLS regressions of the determinants of real yields (Panel A) and volatilities of real yields (Panel B). Real yields, are from Chernov and Mueller (2012) and Gürkaynak et al. (2010). Real yield volatilities are estimated from a GARCH(1,1). Explanatory variables include inflation belief dispersion (Dispersion) and the mean of the inflation forecast (Mean Inflation). The t-statistics are Newey-West corrected. The mean and dispersion of monthly inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: Q1 1978 - Q4 2011.

<table>
<thead>
<tr>
<th></th>
<th>3 Month</th>
<th>6 Month</th>
<th>1 Year</th>
<th>2 Year</th>
<th>3 Year</th>
<th>5 Year</th>
<th>7 Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>0.017</td>
<td>0.017</td>
<td>0.016</td>
<td>0.018</td>
<td>0.017</td>
<td>0.003</td>
<td>0.006</td>
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<tr>
<td>t-statistics</td>
<td>1.515</td>
<td>1.650</td>
<td>1.926</td>
<td>1.844</td>
<td>0.324</td>
<td>0.452</td>
<td>0.689</td>
</tr>
<tr>
<td>Dispersion</td>
<td>0.170</td>
<td>0.319</td>
<td>0.159</td>
<td>0.314</td>
<td>0.141</td>
<td>0.334</td>
<td>0.301</td>
</tr>
<tr>
<td>t-statistics</td>
<td>0.888</td>
<td>0.877</td>
<td>1.092</td>
<td>0.841</td>
<td>1.216</td>
<td>2.130</td>
<td>3.091</td>
</tr>
<tr>
<td>Mean Inflation</td>
<td>-0.171</td>
<td>-0.178</td>
<td>-0.202</td>
<td>-0.357</td>
<td>-0.367</td>
<td>-0.367</td>
<td>-0.367</td>
</tr>
<tr>
<td>t-statistics</td>
<td>-0.713</td>
<td>-0.788</td>
<td>-0.984</td>
<td>-2.427</td>
<td>-2.635</td>
<td>-2.974</td>
<td>-3.154</td>
</tr>
<tr>
<td>Adj.R2</td>
<td>0.015</td>
<td>0.016</td>
<td>0.018</td>
<td>0.014</td>
<td>0.120</td>
<td>0.177</td>
<td>0.118</td>
</tr>
<tr>
<td>N</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>136</td>
<td>136</td>
<td>136</td>
</tr>
</tbody>
</table>

Panel A: Yields

<table>
<thead>
<tr>
<th></th>
<th>3 Month</th>
<th>6 Month</th>
<th>1 Year</th>
<th>2 Year</th>
<th>3 Year</th>
<th>5 Year</th>
<th>7 Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>t-statistics</td>
<td>-0.226</td>
<td>-0.295</td>
<td>-0.198</td>
<td>-0.265</td>
<td>-0.084</td>
<td>-0.147</td>
<td>1.647</td>
</tr>
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<td>Dispersion</td>
<td>0.187</td>
<td>0.203</td>
<td>0.174</td>
<td>0.189</td>
<td>0.159</td>
<td>0.165</td>
<td>0.095</td>
</tr>
<tr>
<td>t-statistics</td>
<td>3.216</td>
<td>3.010</td>
<td>3.208</td>
<td>2.999</td>
<td>3.198</td>
<td>2.992</td>
<td>2.991</td>
</tr>
<tr>
<td>Mean Inflation</td>
<td>-0.018</td>
<td>-0.017</td>
<td>-0.017</td>
<td>-0.026</td>
<td>-0.026</td>
<td>-0.020</td>
<td>-0.016</td>
</tr>
<tr>
<td>t-statistics</td>
<td>-0.219</td>
<td>-0.225</td>
<td>-0.257</td>
<td>-0.560</td>
<td>-0.607</td>
<td>-0.806</td>
<td>-1.150</td>
</tr>
<tr>
<td>Adj.R2</td>
<td>0.308</td>
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Table 3: Inflation Beliefs Dispersion and Real Yields II. The table reports results from OLS regressions of the determinants of real yields (Panel A) and volatilities of real yields (Panel B). Real yields, are computed from the CRSP Risk-Free Rates File (1- and 3-months nominal yields based on bid/ask average prices), the Fama-Bliss Discount Bond File (nominal yields with 1 to 5 years to maturity based on artificial discount bonds) and realized inflation (based on the CPI). Real yield volatilities are estimated from a GARCH(1,1). Explanatory variables include inflation belief dispersion (Dispersion) and the mean of the inflation forecast (Mean Inflation). The t-statistics are Newey-West corrected. The mean and dispersion of monthly inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: January 1978 - December 2011.

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Table 4: Inflation Beliefs Dispersion and Nominal Yields. The table reports results from OLS regressions of the determinants of nominal yields (Panel A) and volatilities of nominal yields (Panel B). Nominal yields are from the CRSP Risk-Free Rates File (1- and 3-months nominal yields based on bid/ask average prices) and the Fama-Bliss Discount Bond File (nominal yields with 1 to 5 years to maturity based on artificial discount bonds). Nominal yield volatilities are estimated from a GARCH(1,1). Explanatory variables include inflation belief dispersion (Dispersion) and the mean of the inflation forecast (Mean Inflation). The t-statistics are Newey-West corrected. The mean and dispersion of inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: January 1978 - December 2011.

### Panel A: Yields

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Figure 1: Average Nominal Yields and Instantaneous Nominal Yield Volatilities - Differences in Beliefs Example
Figure 2: Average Yields and Volatilities with Different Disagreement - Differences in Beliefs Example

Real Yield Curve

Local Volatility of Real Yields

Nominal Yield Curve

Local Volatility of Nominal Yields
Figure 3: Real Yields as a Function of the Sharing Rule - Differences in Beliefs Example

Habit ($\Delta \bar{x} = 1\%$ and $\Delta \kappa = -0.3$)
Figure 4: Feedback Effect
Figure 5: **Mean Term Structure of Nominal and Real Yields and Yield Volatilities.** This figure shows the mean of nominal and real yields and yield volatilities for various maturities. Yield volatilities are computed from a GARCH(1,1). The figure shows the means for the entire sample and for a sample that excludes the high inflation period (with start date: April 1981). The monthly yield data are from the CRSP Risk-Free Rates File (1- and 3-month yields based on bid/ask average prices) and the Fama-Bliss Discount Bond File (yields with 1 to 5 years to maturity based on artificial discount bonds). Real yields are computed by subtracting the mean inflation forecast from nominal yields. The mean of monthly inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: January 1978 - December 2011.
Figure 6: **Mean Inflation Beliefs and Inflation Beliefs Dispersion.** This figure shows mean inflation forecasts and inflation based on the CPI (top plot) and inflation forecast dispersions (bottom plot), the standard deviation of forecasts around the mean forecast —based on Michigan Surveys of Consumers— with NBER recessions as gray shaded areas. Monthly inflation is plotted one year ahead (until July 2012). The mean and dispersion of monthly inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: January 1978 - December 2011.
Figure 7: Inflation Beliefs Dispersion and the Term Structure of Nominal and Real Yields. This figure shows the mean of nominal and real yields for various maturities sorted on beliefs dispersion (Michigan Surveys of Consumers) into five buckets. Nominal and real curves are shown for the entire sample and for a sample that excludes the high inflation period (with start date: April 1981). The monthly yield data are from the CRSP Risk-Free Rates File (1- and 3-month yields based on bid/ask average prices) and the Fama-Bliss Discount Bond File (yields with 1 to 5 years to maturity based on artificial discount bonds). Real yields are computed by subtracting the mean inflation forecast from nominal yields. Monthly dispersion of inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: January 1978 - December 2011.
Figure 8: Inflation Beliefs Dispersion and the Term Structure of Nominal and Real Yield Volatilities. This figure shows the mean of nominal and real yield volatilities for various maturities sorted on beliefs dispersion (Michigan Surveys of Consumers) into five buckets. Nominal and real volatility curves are shown for the entire sample and for a sample that excludes the high inflation period (with start date: April 1981). The monthly yield data are from the CRSP Risk-Free Rates File (1- and 3-month yields based on bid/ask average prices) and the Fama-Bliss Discount Bond File (yields with 1 to 5 years to maturity based on artificial discount bonds). Real yields are computed by subtracting the mean inflation forecast from nominal yields. Yield volatilities are computed from a GARCH(1,1). Monthly dispersion of inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: January 1978 - December 2011.
References


