How Does A Firm’s Default Risk Affect Its Expected Equity Return?

Kevin Aretz∗

Abstract

In a standard representative agent economy, a firm’s default probability and its expected equity return are non-monotonically related, potentially explaining the surprising association between these two variables in recent empirical studies. Although changes in default risk induced through the expected asset payment and debt correlate positively with changes in the expected equity return, changes in default risk induced through asset volatility have an ambiguous impact on the expected equity returns of highly distressed firms. For highly distressed firms, an increase in asset volatility benefits equityholders through a higher chance of obtaining a positive payment, but it can also hurt them through probability mass being shifted from more desirable to less desirable states of nature. Preliminary empirical evidence supports the main implications of the model economy.

Keywords Default risk premium, asset pricing, macroeconomic conditions

JEL Classification G11, G12, G15

This version December 4, 2009

∗The author is at Lancaster University Management School. Address for correspondence: Kevin Aretz, Department of Accounting & Finance, Lancaster University Management School, Bailrigg, Lancaster LA1 4YX, UK, tel.: +44(0)1524-593 402, fax.: +44(0)1524- 847 321, e-mail: <k.aretz@lancaster.ac.uk>. I thank Martin Widdicks for helpful comments and suggestions. I gratefully acknowledge research funding from the Lancaster University Small Grant Scheme.
How Does A Firm’s Default Risk Affect Its Expected Equity Return?

Abstract

In a standard representative agent economy, a firm’s default probability and its expected equity return are non-monotonically related, potentially explaining the surprising association between these two variables in recent empirical studies. Although changes in default risk induced through the expected asset payment and debt correlate positively with changes in the expected equity return, changes in default risk induced through asset volatility have an ambiguous impact on the expected equity returns of highly distressed firms. For highly distressed firms, an increase in asset volatility benefits equityholders through a higher chance of obtaining a positive payment, but it can also hurt them through probability mass being shifted from more desirable to less desirable states of nature. Preliminary empirical evidence supports the main implications of the model economy.

Keywords Default risk premium, asset pricing, macroeconomic conditions

JEL Classification G11, G12, G15

This version December 4, 2009
1. Introduction

A large number of academics and practitioners seem to believe that highly distressed firms (i.e., firms with a high chance of being unable to fulfill their debt obligations) should reward equity investors with high expected equity returns compared to safer firms (e.g., Fama and French, 1995; Dichev, 1998; Griffin and Lemmon, 2002). Often, this belief is motivated by the theoretical studies of Rietz (1988) and Barro (2005), who show that firms with low equity payoffs during major economic crises are risky to hold for equity investors, and that their shares should thus attract high expected returns in equilibrium. As the majority of firms declare bankruptcy in economic recessions, “the idea [of a positive link between a firm’s default risk and its expected equity return] has a certain plausibility” (Campbell, Hilscher, and Szilagy, 2008, p.2900). Notwithstanding these arguments, as the theoretical studies cited above focus exclusively on an all-equity firm, they can offer little insights into the exact relation between default risk (or its determinants) and the expected equity return. Clearly, a firm’s default risk should not only depend on how strongly its asset payments are correlated with consumption, but also on more firm-specific factors, like, e.g., its profitability, leverage ratio or business risk.

In this study, I analyze the relation between default risk and the expected equity return within the classical representative agent economy outlined in Rubinstein (1976) and Huang and Litzenberger (1988). In this framework, which can be seen as a discrete-time, equilibrium version of the framework employed by Black and Scholes (1973), the equity payoff equals the maximum of the asset payment minus debt and zero. Both the future asset payment and future consumption follow a multivariate log-normal distribution. There is a representative agent, and her preferences can be described with time-additive power utility functions. As a result, a firm’s default risk depends on the expectation and variance of the natural logarithm of the future asset payment and on the future debt re-payment.\footnote{While slightly misleading, I will from here onwards refer to the first two factors as the expected asset payment and the asset volatility, respectively.} Consistent with intuition, default risk decreases in the expected asset payment and increases in the debt re-payment. Moreover, default risk increases in asset volatility for firms with a below 50% default probability, but decreases in asset volatility for firms with an above 50% default probability. The ability to decrease default risk through increasing asset volatility when being close to bankruptcy is well known in the literature, and is generally referred to as the ‘risk-shifting problem’ (e.g., Mason and Merton, 1985).

Assuming a positive correlation between asset payment and consumption, my main findings on the impact of default risk on the expected equity return are as follows. First, if default risk does not depend on asset volatility, i.e., if the asset volatility of all firms is equal to a positive constant, then changes in default risk induced through either or both the expected asset payment and the debt re-payment are positively related to the expected equity return. The strength of this relationship depends on a firm’s absolute level of default risk, e.g., the expected equity return of a highly solvent firm remains almost unaffected by changes in default risk.
induced through these two factors. Second and more interestingly, assume that default risk also depends on asset volatility. In this case, changes in default risk induced through asset volatility associate positively with changes in the expected equity return for relatively solvent firms, i.e., those with a default probability below 50%. On the other hand, the expected equity return can \textit{both increase or decrease} in default risk induced through asset volatility for relatively insolvent firms, i.e., those with a default probability above 50%. More specifically, the relation is negative (positive) for firms with a default probability closer to 50% (100%).

What is the economic intuition behind these findings? The sign and magnitude of the association between asset volatility and equity price depends on two effects. First, a change in asset volatility can affect equityholders through its impact on their chance of obtaining a positive payout. Second, a change in asset volatility can also affect risk-averse equityholders through a shift of equity payments across states of nature. For example, assume that the debt re-payment is equal to the median of the log-normal asset payment distribution.\textsuperscript{2} A higher asset volatility "flattens" the asset payment distribution, and therefore renders low equity payments less likely and high equity payments more likely. However, if asset payment and consumption are positively related, then marginal utility is on average higher in states of nature with low equity payments. As a result, equity investors dislike this effect of higher asset volatility. For firm with a default probability slightly greater than 50%, an increase in asset volatility leads to a low decrease in default risk, but a large shift of equity payments from more desirable to less desirable states. In total, the effect on the equity price is positive but small. In contrast, the higher asset volatility leads to a more pronounced increase in the expected equity payment, so that the expected equity return (i.e., the expected equity payment over the equity price) increases in asset volatility. If a firm’s default probability is closer to 100%, an increase in asset volatility does not shift equity payments from more to less desirable states. Hence, the effect on the equity price is larger, leading to a negative relation between asset volatility and expected equity return.

Surprisingly little is required to show that expected equity returns may not monotonically relate to default risk, e.g., my findings hold in the absence of market imperfections (e.g., George and Hwang, 2008) or investor irrationality. Moreover, my theoretical analysis focuses on only one possible density function. As under other density functions the two effects considered above should also be present, it cannot be ruled out that expected equity returns could also decrease in default risk for firms with a default probability below 50%.

My theoretical findings might explain why empirical studies often fail to discover a significant default risk premium. More specifically, although early studies often claim that default risk helps to explain average equity returns, more recent work has cast a shadow of doubt on this evidence. For example, the evidence of Chan, Chen, and Hsieh (1985), Chen, Roll, and Ross (1986) and He and Ng (1994), which reveals that the average equity return loads on exposure to changes in the yield spread between Aaa- and Baa-rated corporate bond portfolios, \textsuperscript{2}In this case, a change in asset volatility does not affect the firm’s default probability.
can be interpreted as showing that expected equity returns depend on exposure to aggregate default risk. Using a proxy variable derived from Merton’s (1974) contingent claims analysis, Vassalou and Xing (2004) offer further support for this conjecture. However, if firms with high aggregate default risk exposure also exhibit high default risk, then default risk itself might be a priced factor. Similarly, some studies, as, e.g., Queen and Roll (1987), Chan and Chen (1991) and Shumway (2001), illustrate that firm size is negatively related to default risk. The high average equity returns of small firms could therefore be a compensation for default risk (e.g., Fama and French, 1995). Somewhat surprisingly therefore, studies more directly analyzing the link between default risk and average equity returns usually find a flat or even negative relation. Often, the conclusions of these more direct studies are based on the average equity returns of portfolios sorted on default risk. Popular proxies for default risk include: (i) the predictions from a dynamic logit model forecasting bankruptcy (Campbell, Hilscher, and Szilagy, 2008), (ii) S&P credit risk ratings (Avramov, Chordia, Jostova, and Philipov, 2007), (iii) default probabilities obtained from structural credit risk models (Garlappi, Shu, and Yan, 2006; Zhang, 2007) or (iv) accounting-based bankruptcy likelihood measures (Dichev, 1998; Griffin and Lemmon, 2002).

To find out whether my findings can contribute to our understanding of equity return behavior, I run some preliminary empirical tests on the main implications of the model. The main implications of the model are as follows: (i) controlling for asset volatility, average equity returns should increase in the difference between the natural logarithm of debt and the expected asset payment; (ii) controlling for the difference between the natural logarithm of debt and the expected asset payment, average equity returns should increase in asset volatility, if default risk is below 50%. The strength of both relations should increase in default risk. In fact, we should not expect to find any relation for highly solvent firms. If default risk is above 50%, average equity returns can, all else equal, increase or decrease in asset volatility. The average equity returns of a portfolio of firms with default probabilities all above 50% is thus likely to show no association with asset volatility. A last implication of the model is: (iii) the equity return spread between highly distressed and solvent firms, i.e., the default risk premium, should depend on macroeconomic instruments capturing time preferences, the correlation between asset payments and consumption, and expected consumption growth.

As the Merton (1974) default probability is a sufficient statistic for forecasting bankruptcy within the model economy, I use it to sort firms into default risk quintile portfolios. In line with prior studies, the average equity returns of these portfolio only mildly increase in default risk. As predicted, only the equity returns of the highest default risk quintile load positively and significantly on both the difference between the natural logarithm of debt and the expected asset payment and asset volatility. Since even the highest default risk quintile contains mostly firms with a below 50% default probability, I split the firms in this quintile into two new portfolio based on whether their default risk is above or below 50%. Again confirming my hypotheses, the portfolio of firms with a below 50% default probability continues to load positively and significantly on asset volatility, while
the portfolio of firms with an above 50% default probability produces a negative and insignificant coefficient. Finally, I also find evidence that the equity return spread between high and low default risk firms depends to some extent on several lagged macroeconomic variables. Inferences from all tests are based on a modified Politis and Romano (1994) bootstrap to account for the slow-moving nature of the explanatory variables.

The article is organized as follows. In Section 2, I first introduce the theoretical model, and then offer my main theoretical findings. I also present some comparative statistics. Section 3 discusses my data sources and data construction procedures, and the main outcomes from my tests. Section 4 concludes. Proofs are in Appendix A. A short description of the Politis and Romano (1994) bootstrap is in Appendix B.

2. The Model Economy

2.1. Assumptions

Following Rubinstein (1976) and Huang and Litzenberger (1988), I analyze a two-period, discrete-time securities market economy, in which a representative agent exists. The preferences of the representative agent are described by time-additive power utility functions, i.e., utility equals:

\[ u_t(z_t) = \frac{1}{1-B} z_0^{1-B} + \rho \frac{1}{1-B} z_1^{1-B}, \]

where \( u_t(z_t) \) is utility of \( z \) at time \( t \), \( B \) is relative risk aversion, and \( \rho \) is the time-preference parameter. Let \( p_A \) denote the market value of a firm’s assets at time 0, whose random payout at time 1 is \( \tilde{x} \). A firm’s assets can be financed with a combination of equity and debt. Debt gives rise to a fixed re-payment at time 1 denoted by \( k \). Finally, the constant \( C_0 \) and the random variable \( \tilde{C}_1 \) are defined as aggregate consumption at time 0 and 1, respectively. Note that a tilde is used to indicate a random variable.

The random variables \( \tilde{x} \) and \( \tilde{C}_1 \) follow a multivariate log-normal distribution, which implies that \( \ln(\tilde{x}_j) \) and \( \ln(\tilde{C}_1) \) follow a multivariate normal distribution. Let the first moments of \( \ln(\tilde{x}) \) and \( \ln(\tilde{C}_1) \) be defined as \( \hat{\mu}_x \) and \( \hat{\mu}_C \). The variance-covariance matrix of \( \ln(\tilde{x}) \) and \( \ln(\tilde{C}_1) \) can be written as:

\[
\begin{bmatrix}
\hat{\sigma}_x^2 & \kappa \hat{\sigma}_x \hat{\sigma}_C \\
\kappa \hat{\sigma}_x \hat{\sigma}_C & \hat{\sigma}_C^2
\end{bmatrix},
\]

where \( \kappa \) denotes the correlation coefficient between \( \ln(\tilde{x}) \) and \( \ln(\tilde{C}_1) \). For notational convenience, it is useful to define \( \tilde{z} \) as \( \ln(\tilde{x}) \) and to define \( \tilde{y} \) as \( \ln[(\tilde{C}_1/C_0)^{-B}] \). I will often refer to the exponential of \( \tilde{y} \) as the marginal rate of substitution or the (stochastic) discount factor. Obviously, \( \tilde{z} \) and \( \tilde{y} \) also follow a multivariate normal distribution. While the first moment of \( \tilde{z} \), denoted by \( \mu_x \), equals \( \hat{\mu}_x \), the first moment of \( \tilde{y} \), denoted by \( \mu_C \), now becomes equal to \( B(C_0 - \hat{\mu}_C) \). Moreover, the variance-covariance matrix between \( \tilde{z} \) and \( \tilde{y} \) is:
\[
\begin{bmatrix}
\sigma_x^2 & \sigma_{x,C} \\
\sigma_{x,C} & \sigma_C^2
\end{bmatrix}
= \begin{bmatrix}
\hat{\sigma}_x^2 & -B\kappa \hat{\sigma}_x \hat{\sigma}_C \\
-B\kappa \hat{\sigma}_x \hat{\sigma}_C & B^2 \hat{\sigma}_C^2
\end{bmatrix}.
\]

2.2. Default Probability

The payment on the equity claim at time 1 equals the maximum of the payment on the assets minus the debt re-payment and zero, i.e., \( \check{x}_E = \max(\check{x} - k, 0) \). A default occurs, if the realization of the asset payment is below the debt re-payment, implying that at time 0 the firm’s default probability (dr) equals:

\[
dr = \text{Prob}(\check{x} < k) = \text{Prob}\left[ \frac{\check{x} - \hat{\mu}_x}{\sigma_x} < \frac{\ln k - \hat{\mu}_x}{\sigma_x} \right] = N\left[ \frac{\ln k - \mu_x}{\sigma_x} \right],
\]

where \( N[.] \) stands for the cumulative standard normal density function. Formula [1] reveals that a firm’s default probability is a highly non-linear function of a firm’s expected asset payment (\( \mu_x \)), its debt re-payment (\( k \)), and its asset volatility (\( \sigma_x \)). As a result, default risk depends on expected profitability, leverage, and risk. Consistent with intuition, none of these variables influence the marginal distribution of consumption at time 1.

We can determine the impact of each of these factors on default risk through taking the partial derivative of \( dr \) with respect to \( \mu_x \), \( k \) and \( \sigma_x \):

\[
\frac{\partial dr}{\partial \mu_x} = -\frac{1}{\sigma_x} n \left[ \frac{\ln k - \mu_x}{\sigma_x} \right] < 0,
\]
\[
\frac{\partial dr}{\partial k} = \frac{1}{k \sigma_x} n \left[ \frac{\ln k - \mu_x}{\sigma_x} \right] > 0,
\]
\[
\frac{\partial dr}{\partial \sigma_x} = -\frac{\ln k - \mu_x}{\sigma_x^2} n \left[ \frac{\ln k - \mu_x}{\sigma_x} \right],
\]

where \( n \) is the density function of the standard normal distribution. An increase in the expected asset payment or a decrease in the debt re-payment shifts probability mass from states of nature in which the firm is unable to re-pay its debt (i.e., states in which \( \check{x} < k \)) to states of nature in which the firm can re-pay its debt (i.e., states in which \( \check{x} > k \)). A firm’s default probability therefore decreases in the expected asset payment and increases in the debt re-payment. Interestingly, asset volatility has an ambiguous impact on default risk. If \( (\ln k - \mu_x) < 0 \) and, in turn, \( k < e^{\mu_x} = \text{median}(\check{x}) \), then default risk is below 50% and increases in asset volatility. However, if \( (\ln k - \mu_x) > 0 \), then default risk is above 50% and decreases in asset volatility.

2.3. Expected Payouts and Market Prices

The expected equity payment, i.e., the expected maximum of the net asset payment and zero, equals:

\[
E[\check{x}_E] = E[\max(\check{x} - k, 0)] = e^{\mu_x + \frac{1}{2} \sigma_x^2} N\left[ \frac{\mu_x + \sigma_x^2 - \ln k}{\sigma_x} \right] - k N\left[ \frac{\mu_x - \ln k}{\sigma_x} \right].
\]
As \( \hat{x} \) is log-normally distributed, the closed-form solution for the expected equity payment can easily be derived using the formula for the truncated mean of log-normal random variables (e.g., Ingersoll, 1987). We can interpret the first term on the right-hand as the expectation of \( \hat{x} \) with asset payments below \( k \) set equal to zero. The second term can be viewed as the firm’s debt re-payment times its survival probability. The second term in the product is a firm’s survival probability, as, first, the survival probability equals one minus the default probability, and as, second, the properties of the normal distribution imply that:

\[
1 - N[(\ln k - \mu_x)/\sigma_x] = N[(\mu_x - \ln k)/\sigma_x].
\]

Following from the usual first-order conditions (e.g., Lucas, 1978; Brock, 1982), the market price of the equity fulfills:

\[
p_E = \rho E \left[ \max(\hat{x} - k, 0) \left( \frac{C_1}{C_0} \right)^{-B} \right].
\]

Under our distributional assumptions, Rubinstein (1976) and Huang and Litzenberger (1988) show that the closed-form solution of the expectation equals:

\[
\rho e^{\mu_C + \frac{1}{2} \sigma_C^2} \left[ e^{\mu_x + \frac{1}{2} (\sigma_x^2 + 2\sigma_x \sigma_C + \sigma_C^2)} N \left( \frac{\mu_x + \sigma_x \sigma_C + \sigma_C^2 - \ln k}{\sigma_x} \right) - k N \left( \frac{\mu_x + \sigma_x \sigma_C - \ln k}{\sigma_x} \right) \right].
\]

As the asset price and the gross riskfree rate are equal to \( \rho e^{\mu_C + \frac{1}{2} (\sigma_C^2 + 2\sigma_x \sigma_C + \sigma_C^2)} \) and \( (\rho e^{\mu_C + \frac{1}{2} \sigma_C^2})^{-1} \), respectively, one can transform formula [3] into its classical form first stated by Black and Scholes (1973).\(^3\)

By definition, the expected equity return equals the expected equity payment over the equity price. In other words, it is the expectation of ‘raw’ equity payments over the expectation of ‘properly discounted’ equity payments, where the discount factor is the marginal rate of substitution. We can derive an analytic formula for the expected equity return through dividing equation [2] by the equity price:

\[
1 = E[\hat{R}_E \rho (C_1/C_0)^{-B}] = E[\hat{R}_E] E[\rho (C_1/C_0)^{-B}] + \text{cov}(\hat{R}_E, \rho (C_1/C_0)^{-B}),
\]

where \( \hat{R}_E \) is the equity return, i.e., the equity payment over the equity price. Re-arranging:

\[
E[\hat{R}_E] = \frac{1 - \text{cov}(\hat{R}_E, \rho (C_1/C_0)^{-B})}{E[\rho (C_1/C_0)^{-B}]}.
\]

Formula [4] reveals that, as the expectation of the marginal rate of substitution does not depend on firm-specific

\(^3\)The market value of the assets (\( p_A \)) is:

\[
\rho E[\hat{x}(C_1/C_0)^{-B}] = \rho E[e^{\ln z + \ln (C_1/C_0)^{-B}}] = \rho E[e^{\hat{z} + \hat{y}}] = \rho e^{E[\hat{z} + \hat{y}] + \frac{1}{2} \sigma_{\hat{z} + \hat{y}}^2}.
\]

As defined, the expectation of \( \hat{z} \) and \( \hat{y} \) are \( \mu_x \) and \( \mu_C \). The variance of the sum is \( \sigma_{\hat{z} + \hat{y}}^2 = 2\sigma_x \sigma_C + \sigma_C^2 \). We can price the riskfree rate through: \( 1 = \rho E[(C_1/C_0)^{-B}] (1 + r_f) \), where \( (1 + r_f) \) is the non-stochastic gross riskfree rate of return. Noting that the expectation equals \( e^{\mu_C + \frac{1}{2} \sigma_C^2} \) and re-arranging produces the desired outcome.
parameters, firm characteristics can only influence the expected equity return through having an impact on the covariance between the equity return and the marginal rate of substitution. If the representative agent is risk-averse \((B > 0)\) and asset payment and consumption are positively correlated \((\kappa > 0)\), then the co-movement between equity return and marginal rate of substitution is negative. Hence, if a change in default risk decreases the covariance term (i.e., turns it more negative), the expected equity return increases. In contrast, if a change in default risk increases the covariance term, the expected equity return decreases.

2.4. Default Risk Drivers, Expected Equity Payment and Equity Price

In this section, I analyze the relations between the drivers of default risk (i.e., the expected asset payment, the debt re-payment, and asset volatility) and the two components making up the expected equity return. Not only will these outcomes be of crucial importance for the derivation of the relation between default risk and the expected equity return, they also shed some light on the intuition behind my subsequent propositions. More specifically, they help to understand how changes in default risk shift the probability distribution of equity payments, and how equity investors react to these shifts of probability mass.

I start with the relationships between the drivers of default risk and the expected equity payment. We can determine the impact of a change in the expected asset payment on the expected equity payment from:

\[
\frac{\partial E[\tilde{x}_E]}{\partial \mu_x} = e^{\mu_x + \frac{1}{2} \sigma_x^2} N \left[ \frac{\mu_x + \sigma_x^2 - \ln k}{\sigma_x} \right] > 0.
\]

An increase in the expected asset payment shifts probability mass from states in which the firm has to declare bankruptcy to states in which it can honor its debt obligations. Also, one of the following two effects occurs: (1) the probability of low equity payments decreases, while that of high equity payments increases; or (2) the probability of all equity payments increases. Which of the two effects occurs depends on the magnitude of the debt re-payment. In both cases, the expected equity payment must obviously increase.

Turning to the partial derivative of the expected equity payment with respect to the debt re-payment:

\[
\frac{\partial E[\tilde{x}_E]}{\partial k} = -N \left[ \frac{\mu_x - \ln k}{\sigma_x} \right] = -\text{survival probability} < 0.
\]

As the firm’s survival probability converges to one, equityholders obtain a strictly positive equity payment in all future states of nature. In this case, the expected equity payment decreases one to one with an increase in debt. In contrast, if the survival probability is strictly smaller than one, then an increase in \(k\) is not subtracted from the equity payment in all future states of nature. As a result, the expected equity payment decreases at a smaller rate than the increase in debt.
The partial derivative of the expected equity payment with respect to asset volatility is:

\[ \frac{\partial E[\hat{x}E]}{\partial \sigma_x} = \sigma_x e^{\mu_x + 0.5 \sigma^2} N \left[ \frac{\mu_x + \sigma^2 - \ln k}{\sigma_x} \right] + kn \left[ \frac{\mu_x - \ln k}{\sigma_x} \right] > 0. \]

A higher asset volatility increases the chance of observing both high and low realizations of the asset payment by shifting probability mass from the center of the distribution to the tails. Still, for \( k \to 0 \), the higher asset volatility increases the expected equity payment, as \( E[\hat{x}] = e^{\mu_x + (1/2)\sigma^2} \). As the debt re-payment truncates the asset payment probability density from the left and therefore first eliminates the higher chance of lower asset payments realizations, the expected equity payout increases in asset volatility.

I now turn to the relations between the drivers of default risk and the equity price. These relations can be different from those based on the Black and Scholes (1973) formula. Black and Scholes (1973) assume that the market value of assets and the risk-free rate are exogenous and then derive a formula for the market value of a claim on the firm’s assets. This formula depends directly on the asset value and the risk-free rate. Although it is obvious that the market value of the claim must therefore also depend on the factors driving the asset value and the risk-free rate (which could overlap with factors explicitly explaining the market value of the claim), Black and Scholes (1973) cannot identify these factors in their framework. For example, in contrast to our closed-form solution, their closed-form solution for the equity price does not explicitly rely on the expected asset payment. As a result, we cannot interpret the partial derivative of the Black and Scholes (1973) formula with respect to any of its exogenous factors as the change in the equity value with respect to the factor. Imposing stronger assumptions, I can identify the factors driving the asset value and the risk-free rate. We can thus interpret the following partial derivatives as the change in equity value with respect to the default risk drivers.

The partial derivative of the equity price with respect to the expected asset payment is:

\[ \frac{\partial p_E}{\partial \mu_x} = \rho e^{\mu_x + \mu_C + \frac{1}{2} \sigma^2 + 2\sigma_{x,C} + \sigma_C^2} N \left[ \frac{\mu_x + \sigma_{x,C} + \sigma_C^2 - \ln k}{\sigma_x} \right] > 0. \]

Again, a higher expected asset payment increases the chance that equityholders obtain a positive equity payment, i.e., it decreases the default probability. However, as a second effect, an increase in the expected asset payment also renders relatively low equity payments less likely and relatively high equity payments more likely. If asset payment and consumption are positively correlated, then equityholders value low equity payments on average more positively than high equity payments. The reason for this that, for \( \kappa > 0 \), a high equity payoff predicts a ‘prosperous’ state of nature with high consumption, and a low equity payoff predicts a ‘poor’ state of nature with low consumption. In turn, risk-averse agents appreciate additional wealth more in ‘poor’ than in ‘prosperous’ states of nature. More technically, for \( \kappa > 0 \), the equity payoff is negatively related to the conditional expectation of the marginal rate of substitution. As a result, equityholders suffer to some extent from this shift of probability
Turning to the partial derivative of the equity price with respect to the debt re-payment:

\[ \frac{\partial p_E}{\partial k} = -\rho e^{\mu C} + \frac{1}{2} \sigma_x^2 N \left( \frac{\mu_x + \sigma_x + \sigma^2 \ln k}{\sigma_x} \right) < 0. \]

An increase in the debt re-payment increases the default probability and lowers equity payments in all other states of nature. As a result, it decreases the equity price.

An especially interesting partial derivative of the equity price is that with respect to asset volatility:

\[ \frac{\partial p_E}{\partial \sigma_x} = \rho e^{\mu C} + \frac{1}{2} \sigma_x^2 \left[ (\sigma_x - Bk\sigma_C) e^{\mu_x + \frac{1}{2} [\sigma_x^2 + 2\sigma_x \sigma_C]} N \left( \frac{\mu_x + \sigma_x + \sigma^2 \ln k}{\sigma_x} \right) + kn \left( \frac{\mu_x + \sigma_x + \sigma^2 \ln k}{\sigma_x} \right) \right]. \] (5)

In the Black and Scholes (1973) model, asset volatility and the equity price are always positively related. In my framework, this relation can be positive or negative, as can be seen from the following lemma:

**Lemma 1:** The partial derivative of equity value \((p_E)\) with respect to asset volatility \((\sigma_x)\) is positive, if and only if \(Bk\sigma_C < \sigma_x\) or \(Bk\sigma_C > \sigma_x\) and \((\mu_x - \ln k)/\sigma_x < \theta^*\), where \(\theta^*\) is a strictly positive boundary and therefore a value for which the default probability is below 50%. In contrast, the partial derivative is negative, if and only if \(Bk\sigma_C > \sigma_x\) and \((\mu_x - \ln k)/\sigma_x > \theta^*\).

**Proof:** See Appendix A.

An increase in asset volatility renders low and high realizations of the asset payment more likely, but decreases the probability of intermediate realizations. If default risk is above 50% and the debt re-payment is high, then a higher asset volatility is good news for equityholders, as they are only interested in asset payments greater than the debt re-payment, which are now more likely. For a lower debt re-payment with default risk still above 50%, equityholders also benefit from a higher chance of obtaining high equity payments, but they also suffer from a lower chance of obtaining low equity payments. Again, if consumption and asset payment are positively correlated, then equityholders value low equity payoffs on average more positively than high equity payments. If the default probability is below 50%, an increase in asset volatility not only shifts equity payments across states of nature, but also decreases the chance of obtaining a positive equity payoff. In extreme cases, these two effects can create a negative relation between equity price and asset volatility. However, as we would normally...
expect \( B < 1 \) and \( \hat{\sigma}_C < \sigma_x \), a negative partial derivative is unlikely in practice.

For convenience, I list the relations between the drivers of default risk and default risk and those between the drivers of default risk and the components of the expected equity return in the following figure:

\[
\text{Default risk} \downarrow \leftarrow \mu_x \uparrow \rightarrow \begin{cases} E[\tilde{x}_E] \uparrow \rightarrow E[\tilde{R}_E] \,(?) \end{cases} \quad \text{when } \mu_x \uparrow \rightarrow E[\tilde{R}_E] \,(?) \\
\text{Default risk} \uparrow \leftarrow k \uparrow \rightarrow \begin{cases} E[\tilde{x}_E] \downarrow \rightarrow E[\tilde{R}_E] \,(?) \end{cases} \quad \text{when } k \uparrow \rightarrow E[\tilde{R}_E] \,(?) \\
\text{Default risk} \uparrow \leftarrow \sigma_x \uparrow \rightarrow \begin{cases} E[\tilde{x}_E] \uparrow \rightarrow E[\tilde{R}_E] \,(?) \end{cases} \quad \text{when } \sigma_x \uparrow \rightarrow E[\tilde{R}_E] \,(?)
\]

If default risk is below \( \theta^* \) and \( Bk\hat{\sigma}_C > \sigma_x \), the expected equity return increases in asset volatility. As, if the default probability is below 50\%, default risk also increases in asset volatility, there is a positive relation between default risk and the expected equity return in this case. In other cases, my conclusions so far do not allow me to derive the relation between default risk and the expected equity return.

In general, the impact of a change in default risk on the expected equity return depends on its impact on the expected equity payment and the equity price. We have seen that there are two effects. First, a change in default risk affects equityholders’ chance of obtaining a positive payout, which influences both the expected equity payment and the equity price. Second, a change in default risk shifts equity payment probability mass across states of nature. As an example, if \( E[\tilde{R}_E] > 1 \), a shift from low to high equity payments must always strongly increase the expected equity payment. In contrast, the increase in the equity price must be relatively lower, first because the conditional expectation of the discount factor is on average below one, but also because the conditional expectation of the discount factor decreases in the equity payment. In the end, the ultimate question is by how much lower the impact on the equity price is. As I have made the assumption that \( E[\tilde{R}_E] > 1 \), which implies \( E[\tilde{x}_E] > p_E \), an in absolute terms lower increase in the equity price than in the expected equity payment could still increase or decrease the expected equity return.

2.5. Default Risk Premium

Plugging the closed-form solutions for the expected equity payment and the equity price into the definition of the expected equity return, the expected equity return can be stated as:

\[
E[\tilde{R}_E] = \frac{E[\tilde{x}_E]}{p_E} = \frac{1}{pe^{\mu C} + \frac{1}{2} \hat{\sigma}^2 C} e^{\mu_x + \frac{1}{2} \sigma^2_x} N \left\{ \frac{\mu_x + \sigma^2_x - \ln k}{\sigma_x} \right\} - k N \left\{ \frac{\mu_x - \ln k}{\sigma_x} \right\} - \ln \left( \frac{\sigma_{x,C} + \sigma_x^2 - \ln k}{\sigma_x } \right) - k N \left\{ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right\}.
\]  

\[ (6) \]
Formula [6] reveals that, if asset payments are not correlated with consumption, i.e., if \( \kappa = 0 \), then the expected equity return reduces to the inverse of \( (\rho e^{\mu_C + \frac{1}{2} \sigma_C^2}) \). In footnote [3], we have seen that this inverse equals the gross risk-free rate of return. In a similar way, if the representative agent is risk-neutral, i.e., if \( B = 0 \), then the expected equity return reduces to \( (1/\rho) \), which is the inverse of the time preference parameter. In both special cases, the expected equity return does no longer depend on any of the three default risk drivers. Risk-aversion (or the less plausible case of risk-seeking) and a non-zero correlation between asset payment and consumption are therefore necessary conditions for a default risk premium to exist.

More generally, let us initially assume that default risk is exclusively driven by the expected asset payment and the debt re-payment. In other words, my assumption is that the asset volatilities of firms are all equal to a fixed positive constant. In this case, the following proposition describes the relation between default risk and the expected equity return:

**Proposition 1:** Assume a positive covariance between asset payment and consumption, i.e., \( \kappa > 0 \), and no variation in asset volatility across firms, i.e., \( \sigma_{x,i}^2 = a \forall i = 1, 2, \ldots, N \), where \( a \) is a strictly positive constant and \( N \) is the number of firms in the economy. Then:

(i) an increase in the expected asset payment \( (\mu_x) \) decreases both default risk and the expected equity return \( (E[\tilde{R}_E]) \), and vice versa. As a result, variation in default risk induced through the expected asset payment is positively related to the expected equity return.

(ii) an increase in the debt re-payment \( (k) \) increases both default risk and the expected equity return \( (E[\tilde{R}_E]) \), and vice versa. As a result, variation in default risk induced through the debt re-payment is positively related to the expected equity return.

(iii) an increase in the difference between the natural logarithm of the debt re-payment and the expected asset payment \( (\ln k - \mu_x) \) increases both default risk and the expected equity return \( (E[\tilde{R}_E]) \), and vice versa. As a result, variation in default risk induced through changes in both the expected asset payment and the debt re-payment is positively related to the expected equity return.

**Proof:** See Appendix A.

A concrete example should shed more light onto the intuition behind proposition [1]. Let us assume that the relative risk aversion \( (B) \) and the time-preference parameter \( (\rho) \) of the representative agent equal 0.01 and 0.95, respectively. Moreover, the expectation and standard deviation of the natural logarithm of consumption at time 1 (\( \hat{\mu}_C \) and \( \hat{\sigma}_C \), respectively) are set to 0 and 0.15, respectively. The correlation between the natural logarithm of the asset payment and the natural logarithm of consumption \( (\kappa) \) is 0.60. There exists a firm with an expected asset payment \( (\mu_x) \) of 0, a debt re-payment \( (k) \) of 0.80, and an asset volatility \( (\sigma_x) \) of 0.25. Recall
that the expected equity payment is simply the expectation of equity payments, while the equity price is the expectation of properly discounted equity payments, i.e., $p_E = E[x_E E[\rho(C_t/C_0)^{-t} | x_E]]$. In figure 1, I plot the marginal density of $\tilde{x}$ and the conditional expectation of the discount factor. An increase in the expected asset payment from 0 to 0.1 leads to the two effects discussed above. First, the probability of obtaining a positive equity payment increases from 82.1% to 90.6%. Second, the probability of equity payments below (1.05-0.80) is now 7.3% lower, while that of equity payments above (1.05-0.80) is 7.3%+8.5%=15.8% higher. Since, for $\kappa > 0$, equity investors value low equity payments more highly than high equity payments, the expected equity payment increases by 0.099, while the equity price increases by only 0.092. Nevertheless, the increase in the equity price is sufficiently large to make the expected equity return decrease.

Re-considering equation [4], we can also interpret proposition [1] as stating that an increase in the expected asset payment or a decrease in the debt re-payment must always decrease the magnitude of the covariance between the equity return and the marginal rate of substitution.

Consistent with intuition, the impact of a change in a firm’s default risk on its expected equity return depends on its absolute level of default risk. In particular, the expected equity returns of highly solvent firms are hardly affected by changes in the expected asset payment or the debt rep-payment, as can be seen from the following corollary:

**Corollary 1:** As the expected asset payment ($\mu_x$) goes to infinity or as the debt re-payment ($k$) goes to zero, the partial derivative of the expected equity return with respect to the asset payment ($\partial E[R_E]/\partial \mu_x$), the partial derivative of the expected equity return with respect to the debt re-payment ($\partial E[R_E]/\partial k$), and the total derivative of the expected equity return ($dE[R_E]$) converge to zero, i.e.:

$$\lim_{\mu_x \to \infty} \frac{\partial E[R_E]}{\partial \mu_x} = \lim_{\mu_x \to \infty} \frac{\partial E[R_E]}{\partial k} = \lim_{\mu_x \to \infty} dE[R_E] = 0,$$

and

$$\lim_{k \to 0} \frac{\partial E[R_E]}{\partial \mu_x} = \lim_{k \to 0} \frac{\partial E[R_E]}{\partial k} = \lim_{k \to 0} dE[R_E] = 0.$$

**Proof:** See Appendix A.

Let us now turn to the less restrictive case in which variation in default risk can also be driven by variation in a firm’s asset volatility. In this situation, the following proposition shows that there is no longer a monotone relation between default risk and the expected equity return:

**Proposition 2:** Assume a positive covariance between asset payment and consumption, i.e., $\kappa > 0$. Then:
(i) for firms with a default probability below 50%, an increase in asset volatility ($\sigma_x$) increases both default risk and the expected equity return ($E[\tilde{R}_E]$), and vice versa. As a result, variation in default risk induced through asset volatility is positively related to the expected equity return.

(ii) for firms with a default probability between 50% and $\gamma$, where $\gamma$ is a percentage between 50% and 100%, an increase in asset volatility ($\sigma_x$) decreases default risk, but increases the expected equity return ($E[\tilde{R}_E]$), and vice versa. As a result, variation in default risk induced through asset volatility is negatively related to the expected equity return.

(iii) for firms with a default probability above $\gamma$, where $\gamma$ is a percentage between 50% and 100%, an increase in asset volatility ($\sigma_x$) decreases both default risk and, at least at some points, the expected equity return ($E[\tilde{R}_E]$), and vice versa. As a result, variation in default risk induced through asset volatility is, at least at some points, positively related to the expected equity return.

Proof: See Appendix A.

In figure 2, I offer an example of a firm whose expected equity return and default risk are negatively related, although the correlation between asset payment and consumption is positive ($\kappa = 0.60$). Parameter values are identical to those used in figure 1, except that the firm’s debt re-payment is now equal to 1.01. Conditional on the expected asset payment being 0 and the asset volatility being 0.25, the firm’s survival probability equals 49.2%. An increase of asset volatility to 0.40 increases the survival probability to 49.5%, i.e., equityholders’ chance of receiving a strictly positive payments increases by 0.30%. In addition, equity payments below (1.35-1.01) are now 10.9% less likely, while equity payments above (1.35-1.01) are now 11.2% more likely. In contrast to the situation in figure 1, the second effect (the re-distribution of equity payment probability mass) is relatively stronger than the first effect (the increase in the survival chance). The increase in asset volatility leads to an increase in the expected equity payment of 0.092, while the equity price only increases by 0.086. All together, the increase in the equity price is sufficiently small, so that the expected equity return increases.

Interpreting the example above in light of equation [4], it should be noted that the increase in the expected equity return must stem from an increase in the magnitude of the covariance between the equity return and the marginal rate of substitution induced through the change in asset volatility.

Although not the main objective of the study, the model economy also produces some interesting relations between the change in the expected equity return induced through a change in default risk (i.e., the default risk discount premium or discount) and macroeconomic variables, as can be seen in the following proposition. To ensure that there is a monotone relation between the expected equity return and default risk, I re-instate the assumption that default risk only depends on the expected asset payment and the debt re-payment.
**Proposition 3:** Assume a positive covariance between asset payment and consumption, i.e., $\kappa > 0$, and no variation in asset volatility across firms, i.e., $\sigma^2_{x,i} = a \forall i = 1, 2, \ldots, N$, where $a$ is a strictly positive constant and $N$ is the number of firms in the economy. Then:

(i) the magnitude of the default risk premium, i.e., the total derivative of the expected equity return with respect to the drivers of default risk, is positively related to expected consumption at time 1 ($E(\ln(\tilde{C}_1))$) and negatively to consumption at time 0 ($C_0$). As a result, the magnitude of the default risk premium is positively related to consumption growth.

(ii) the magnitude of the default risk premium, i.e., the total derivative of the expected equity return with respect to the drivers of default risk, is negatively related to the time preference parameter ($\rho$).

(iii) the difference between the default risk premium, i.e., the total derivative of the expected equity return with respect to the drivers of default risk, at $\kappa > 0$ and at $\kappa = 0$ is greater than zero.

**Proof:** See Appendix A.

Intuitively speaking, an increase in expected consumption at time 1 ($\tilde{\mu}_C$) “flattens” the conditional expectation of the marginal rate of substitution, and a shift of probability mass from low equity payoffs to high equity payoffs is therefore relatively less hurtful for equityholders. In other words, an increase in the expected asset payment or a decrease in the debt re-payment leads to a greater increase in the equity price, which in turn leads to a larger decrease in the expected equity return. A similar reasoning applies to both consumption today ($C_0$) and the time preference parameter ($\rho$). An direct implication of proposition [3] is that we should see a larger default risk premium in economic recessions than in expansions.

2.6. Comparative Statistics

Figures [3] and [4] offer comparative statistics on the relations between default risk induced through either the expected asset payment ($\tilde{\mu}_x$), the debt re-payment ($k$) or asset volatility ($\sigma_x$) and the expected equity return. In my base case scenario, the time-preference parameter ($\rho$) and the risk aversion parameter ($B$) equal 0.05 and 0.95, respectively. My choices for consumption at time 0 ($C_0$) and expected consumption at time 1 ($\tilde{\mu}_C$) are 1 and 2, respectively. The standard deviation of consumption at time 1 ($\tilde{\sigma}_C$) is 1. Turning to the firm-specific parameters, I set both the expected asset payment ($\mu_x$) and the debt re-payment ($k$) equal to 1, and the standard deviation of the asset payment at time 1 ($\sigma_x$) to 1.25. The correlation between asset payment and consumption ($\kappa$) is 0.60. In Figure [3], I concentrate on the default risk premium across different preferences of the representative agent (i.e., $\rho$ and $B$). Figure [4] analyzes the default risk premium under different macroeconomic conditions (i.e., $\tilde{\mu}_C$ and $\kappa$). The four panels in each of the two figures focus on default
risk induced through (i) the expected asset payment, (ii) the debt re-payment, (iii) asset volatility, if default risk is below 50%, and (iv) asset volatility, if default risk is above 50%.

Figure [3] reveals that the magnitude of the expected equity return is positively related to risk aversion ($B$) and negatively to the time preference parameter ($\rho$). More importantly, it also illustrates that the relation between default risk and the expected equity return is convex, at least for high levels of default risk. When default risk is lower, this relation can become concave, as shown in panel C for $B=0.1$ and $\rho=0.95$. Panels A and B suggest that changes in default risk induced through the expected asset payment or the debt re-payment are always positively associated with changes in the expected equity return. However, the spread in the expected equity return varies substantially across the different parameter choices. For low levels of risk aversion, e.g., when $B$ equals 0.01, the difference in the expected equity return between high and low default risk firms is close to 1%. In contrast, when risk aversion increases to 0.1, this difference can shoot up to 15%. Variation in the time-preference parameter has a much weaker impact on the default risk premium, i.e., a decrease of $\rho$ from 0.99 to 0.90 only increases the spread in the expected equity return by 0.5%. A comparison of Panels A and Panel B offers evidence that the expected equity return increases more strongly in default risk induced through the expected asset payment than default risk induced through the debt re-payment.

Panels C and D analyze the relation between default risk induced through asset volatility and the expected equity return. When $k$ is equal to 1, a change in asset volatility can never increase default risk above 50%. As a result, I change the debt re-payment to 3 in panel D, so that the firm’s default probability is always greater than 50%. In Panel C, in which default risk ranges from 0 to 50%, a change in default risk induced through asset volatility always associates positively with a change in the expected equity return. In comparison to Panels A and B, the default risk premium in Panel C is an order of magnitude greater. More concretely, if $B=0.05$ and $\rho=0.95$, an increase in default risk from 1% to 30% generates a default risk premium of almost 5% in Panel C, while an equivalent change in default risk in Panels A and B only leads to a spread in the expected equity return of less than 1%. Similar to before, the default risk premium increases strongly in $B$ and decreases weakly in $\rho$. In Panel D, the relation between default risk and the expected equity return is U-shaped and therefore no longer monotone, i.e., the expected equity returns of firms with relatively low default risk (close to 50%) decrease in default risk, while those of firms with relatively high default risk (closer to 100%) increase in default risk. The non-linearity increases strongly with $B$ and decreases weakly with $\rho$.

Panels A and B of Figure [4] suggest that expected consumption at time 1 ($\hat{\mu}_C$) and correlation between the asset payment and consumption at time 1 ($\kappa$) are positively related to the default risk premium induced through the expected asset payment and/or the debt re-payment. However, the impact of a change in the co-movement between the asset payment and consumption seems to be stronger than that of a change in expected consumption. When default risk is below 50%, we can draw the same conclusions with respect to the relation
between the default risk premium and asset volatility (Panel C). In contrast, when default risk is above 50%, both the initial decline in the expected equity return and its subsequent increase become steeper with both expected consumption and correlation between asset payment and consumption (Panel D).

2.7. Some Take-Aways of the Model for the Real World

The following conclusions can be distilled from the former analysis. First, a firm’s expected equity return depends exclusively on the expectation of the marginal rate of substitution, which is unrelated to firm attributes, and the covariance between its equity return and the marginal rate of substitution. As a result, a change in default risk can only have an impact on the expected equity return through affecting the covariance between the equity return and the marginal rate of substitution. We therefore cannot determine the relation between default risk and the expected equity return unless we know (i) how equity investors value equity payoffs in different states of nature and (ii) how a change in default risk re-distributes equity payments across states of nature. Obviously, this re-distribution effect is likely to vary across firms, and it could even vary across two changes of default risk of equal size experienced by the same firm. My main conclusion is thus that it is unlikely that there exists a monotone relation between default risk and the expected equity return in the real-world.

Second, under strong assumptions on equity investors’ preferences and the joint distribution of asset payment and consumption, I can categorize changes in default risk into those always monotonically related to the expected equity return and others with more complicated associations with the expected equity return. In my case, changes in default risk induced through the expected asset payment, a measure of profitability, and the debt re-payment, a measure of leverage, always relate positively to changes in the expected equity return, if the correlation between asset payment and consumption is positive. In contrast, changes in default risk induced through asset volatility are non-monotonically associated with the expected equity return. In the remainder of the study, I test whether several implications of my model are supported by real-world data.

3. Empirical Tests

3.1. Testable Implications

My theoretical findings generate the following testable implications. First, proposition [1] predicts that average equity returns should, ceteris paribus, increase in the difference between the natural logarithm of the debt re-payment and the expected asset payment. Second, proposition [2] indicates that the average equity returns of relatively solvent firms should, ceteris paribus, increase in asset volatility, while those of relatively distressed firms can, ceteris paribus, increase or decrease in asset volatility. Third, the strengths of these relations should, by corollary [1], increase in a firm’s level of default risk. Finally, proposition [3] suggests that the spread in the
average equity return between a firm with high default risk and one with low default risk should depend upon
macroeconomic instruments capturing expected consumption growth, the time preferences of equity investors
and the correlation between asset payments and consumption in the future.

3.2. Methodology

The implications can be tested using equity portfolios sorted on default risk. I test the first three implications
through regressing the equity portfolio return on lagged proxies for the difference between the natural logarithm
of the debt re-payment and the expected asset payment and the asset volatility. If the implications of the model
economy are correct, then the equity returns of the relatively low default risk portfolios should not significantly
load on the explanatory variables. In contrast, the equity returns of the relatively high default risk portfolios
should load positively and significantly on the difference between the natural logarithm of the debt re-payment
and the expected asset payment. Further, if the high default risk portfolios contain only (or mostly) firms
with default probabilities below 50%, then their equity returns should also load positively and significantly on
asset volatility. On the contrary, if a high default risk portfolio contains only (or mostly) firms with default
probabilities above 50%, then I conjecture that there are a similar amount of firms with a positive and a negative
relation between asset volatility and the expected equity return, so that the net effect of asset volatility on the
average equity return is weaker and potentially insignificant. I test the final implication through regressing the
equity return of a default risk spread portfolio on lagged macroeconomic instruments.

Both the regression models featuring the empirical proxies for the drivers of default risk and those featuring
the macroeconomic variables contain slow-moving instruments, which can bias statistical inferences (e.g., see
Stambaugh, 1999). Goyal and Welch (2008) suggest that an easy way to avoid this bias in statistical inferences
is to use a bootstrap method preserving the autocorrelation structure of the explanatory variables. I follow their
idea in this study, and compute the critical values for t-statistics using a modified Politis and Romano (1994)
bootstrap method. More details are contained in Appendix B.

I am also interested in establishing whether certain risk factors known to price equities can capture the
impact of the drivers of default risk on average equity returns. To this end, I orthogonalize the equity returns
of all individual firms with respect to risks associated with the market, SMB, HML and WML (e.g., Fama
and French, 1993; Carhart, 1997). Firms’ equity returns are orthogonalized as follows. First, I use 48-month
out-of-sample rolling-window regressions of the equity return onto the realizations of the risk factors to obtain a
firm’s conditional beta exposures. Subsequently, I subtract the sum of the conditional beta exposures multiplied
by the realizations of the risk factors from the equity return.
3.3. Data Construction and Data Sources

A firm’s theoretical default probability can be re-written as follows:

\[ dr = N \left[ \ln k - \mu_x \right] = N \left[ \ln k - \ln p_A + \ln p_A - \mu_x \right] = N \left[ \ln k - \ln p_A + \ln \rho + \mu_C + \frac{1}{2} \left( \sigma_x^2 + 2 \sigma_{x,C} + \sigma_C^2 \right) \right], \]

where \( p_A \) equals the market value of the assets, whose closed-form solution is in footnote [3]. Noticing that \( E[\tilde{x}] = e^{\mu_x + \frac{1}{2} \sigma_x^2} \), let \( E[\tilde{r}_A] = \ln E[\tilde{x}_A] / p_A = - (\ln \rho + \mu_C + \sigma_{x,C} + \frac{1}{2} \sigma_C^2) \). Hence:

\[ dr = N \left[ - \ln(p_A/k) - (\ln \rho + \mu_C + \sigma_{x,C} + \frac{1}{2} \sigma_C^2) - \frac{1}{2} \sigma_x^2 \right] = N \left[ - \ln(p_A/k) + (E[\tilde{r}_A] - \frac{1}{2} \sigma_C^2) \right]. \] (7)

When the time-to-maturity (often denoted by \( T \) in other studies) equals one year, the last expression is equivalent to the Merton (1974) default probability (e.g., Vassalou and Xing, 2004). Hence, it seems natural to approximate default risk using a proxy for the Merton (1974) default probability in my empirical tests. To construct the proxy variable, let \( k_{j,t} \) be the book value of short-term debt plus one-half of the book value of long-term debt of firm \( j \) known to investors at time \( t \). Further, let \( r_{f,t} \) be the risk-free rate of return at time \( t \) as approximated through the 3-month Treasury bill yield. According to the Black and Scholes (1973) model, the market value of firm \( j \)'s equity at time \( t \), \( p_{jE,t} \), must fulfill the following condition:

\[ p_{jE,t} = p_{jA,t} N(d_1)_{j,t} - k_{j,t} e^{-r_{f,t}T} N(d_2)_{j,t}, \] (8)

where \( p_{jA,t} \) is the market value of firm \( j \)'s assets at time \( t \) and \( (d_1)_{j,t} = \ln(p_{jA,t}/k_{j,t}) + (r_{f,t} + 0.5 \sigma_{jx,t}^2)T / \sigma_{jx,t} \sqrt{T} \), \( (d_2)_{j,t} = (d_1)_{j,t} - \sigma_{jx,t} \sqrt{T} \), with \( \sigma_{jx,t} \) being the standard deviation of firm \( j \)'s asset value at time \( t \) and \( T \) being the time-to-maturity. I set the time-to-maturity equal to one year. Formula [8] is identical to the formula for the equity value derived from the model economy in equation [3], if we replace \( p_{jA,t} \) and \( r_{F,t} \) with their closed-form solutions shown in footnote [3]. However, one advantage of formula [8] is that it depends on fewer unobservable variables, i.e., it features asset value and asset volatility, but not the time-preference parameter or the risk-aversion parameter. As Merton (1974) also derives a closed-form solution for equity volatility, one could in principle solve a system consisting of the equity value equation above and the equity volatility equation for the asset value and asset volatility. However, as the equity volatility equation includes leverage, this approach generates asset volatility estimates with great time-series variation, which does not align with the observation that equity and debt volatility cluster over time (Crosbie, 1999).

To avoid excessive time-series variation in one firm’s asset volatility estimates, I use an iterative approach which relies on one-year rolling windows of daily equity values, monthly risk-free rates and annual book values of debt (e.g., see Crosbie, 1999; Vassalou and Xing, 2004). Following this approach, a firm’s monthly asset
value and asset volatility are computed as follows. First, I compute a firm’s equity return variance from its daily equity values over all trading days of the prior year, and I use this estimate as my initial guess of the unknown asset volatility. I then employ numerical methods to back out the firm’s asset value from equation [8] for each trading day over the prior year using my initial guess of asset volatility, the daily equity values, the monthly risk-free rates, and the annual book values of short-term and long-term debt. Using the time-series of daily asset values, I re-estimate the firm’s asset volatility. Finally, I repeat the former two steps until asset volatility converges, which usually happens after two or three iterations. Upon converge, I plug my estimates of asset value ($p_A$), asset volatility ($\sigma_x$), the expected asset return ($E[\tilde{r}_A]$), which is approximated through the mean asset return over the prior year, and the debt re-payment into equation [7] to obtain an estimate of the Merton (1974) default probability. I use the numerator in the cumulative density function in equation [7] as my empirical proxy for the natural logarithm of the debt re-payment minus the expected asset payment and the denominator as my empirical proxy for asset volatility.

I form equity portfolios based on the Merton (1974) default probability in a way similar to Fama and French (1993). In particular, in June of each year $t$ I first separate firms with a positive book value of total debt from those with a zero value. I sort the firms with no debt into a portfolio ‘0’. Using firms with positive debt, I then construct quintile breakpoints based on the Merton (1974) default probability. In contrast to Fama and French (1993), I employ the whole universe of firms (and not only NYSE firms) to compute the breakpoints, the reason being that most firms traded on the NYSE have Merton (1974) default probabilities virtually equal to zero. An exact replication of the Fama and French (1993) approach would therefore generate huge differences in the number of firms across the equity portfolios. Based upon the breakpoints, I sort firms into five quintile portfolios, which are labeled as portfolios ‘1’ (low default risk) to ‘5’ (high default risk). The firms in equity portfolio ‘5’ are further split into a portfolio containing only firms with Merton (1974) default probability below 50% (portfolio ‘5a’) and a portfolio containing only firms with Merton (1974) default probabilities above 50% (portfolio ‘5b’). All portfolios are value-weighted and are held from July of year $t$ to June of year $t+1$, at which point portfolios are re-formed following the same algorithm.

Daily equity values and monthly equity returns including re-invested dividends are from CRSP. I obtain the annual book values of short-term and long-term debt from COMPUSTAT. I lag the accounting variables by three months to ensure that they have been known to equity investors at the time. The 3-month U.S. Treasury bill rate is from Factset’s Economic Database. The returns on the market portfolio, SMB, HML and WML are from Kenneth French’s website. My sample period starts in January 1970 and ends in December 2007 and therefore contains 456 monthly observations.
### 3.4. Empirical Evidence

In Table 1, I report time-series averages of several characteristics of the default risk portfolios. These characteristics include the number of firms, the equity return, default risk, the natural logarithm of the debt re-payment minus the expected asset payment and the asset volatility. Panels A and B feature unadjusted equity returns and equity returns orthogonalized with respect to the Fama and French (1993) and Carhart (1997) risk factors, respectively. Starting with Panel A, most portfolios contain an average of around 600 firms, although portfolios ‘5a’ and ‘5b’ contain only 463 and 169 firms (i.e., only 25% of the firms in portfolio ‘5’ exhibit a default probability above 50%). In line with prior studies, average equity returns increase only moderately and insignificantly over the default risk quintile portfolios, i.e., the spread in average equity returns is only 0.14. Given the only weak increase in default risk over the quintile portfolios, it is probably unsurprising that average equity returns are similar. Supporting this claim, the very high default risk firms in portfolio ‘5b’ exhibit an average equity return 0.25% higher than those in portfolio ‘1’. Finally, high default risk firms show both a significantly higher average difference between the natural logarithm of the debt re-payment and the expected asset payment and, with the exception of portfolio ‘0’, a significantly higher average asset volatility.

After the risk adjustment of equity returns, average equity returns decrease to around half of their former magnitude (Panel B). More importantly, they now slightly, but still insignificantly decrease over the default risk portfolios. Notwithstanding these observations, the very high default risk firms in portfolio ‘5b’ still exhibit a much higher average equity return than the firms in the other portfolios. As a result, it seems that the equity pricing risk factors proposed by Fama and French (1993) and Carhart (1997) can, at least not entirely, capture extra compensation for default risk. Other statistics remain similar to before.

Table 2 offers the outcomes from OLS regressions of the equity portfolio returns onto my 2-month lagged proxies for the difference between the natural logarithm of the debt re-payment and the expected asset payment and the asset volatility. Bold numbers are parameter estimates, the numbers in square parentheses are t-statistics, and the two numbers in curly parentheses are the 2.5% and 97.5% critical values of the t-statistic derived from the stationary bootstrap of Politis and Romano (1994). In addition, I show the adjusted $R^2$ of each regression. Panels A and B contain the regressions featuring the unadjusted equity returns and those featuring the equity returns orthogonalized with respect to the risk factors, respectively. My empirical evidence in Panel A strongly suggests that the equity returns of portfolios with moderate levels of default risk are not significantly related to either one of the two explanatory variables, supporting corollary [1]. Only the equity return of portfolio ‘5’ loads both positively and significantly on the difference between the debt re-payment and the expected asset payment and the asset volatility. Although still low in absolute terms, the adjusted $R^2$ of

---

5Firms in portfolio ‘0’ are only required to have a zero book value of total debt on the portfolio formation date. This explains why their average default probability is slightly greater than zero.
the regression featuring portfolio ‘5’ (1%) is far larger than those of the other regressions (around 0%). Given that most firms in portfolio ‘5’ have a below 50% default probability, the signs of both loadings are consistent with propositions [1] and [2].

As a next step, I separately consider the firms in portfolio ‘5’ with a below 50% default probability (portfolio ‘5a’) and those with an above 50% default probability (portfolio ‘5b’). In line with propositions [1] and [2], the equity returns of portfolio ‘5a’ obtain a positive and significant loading on both the difference between the debt re-payment and the expected asset payment and the asset volatility. In comparison to portfolio ‘5a’, the equity returns of portfolio ‘5b’ load more positively on the difference between the debt re-payment and the expected asset payment, but only at a slightly below 90% confidence level. The low significance level is probably driven by the low number of firms in this portfolio and the resulting noise in its equity return. However, more striking is the observation that the equity returns of portfolio ‘5b’ fail to load on its asset volatility, i.e., the coefficient estimate drops from 2.66 or 2.65 for portfolios ‘5’ or ‘5a’ to -0.35 for portfolio ‘5b’, respectively, and the t-statistic drops from 3.16 or 3.29 to -0.41, respectively. In fact, the absolute value of the t-statistic on the asset volatility of portfolio ‘5b’ is the lowest across all portfolios. Again, if there is a roughly equal number of firms whose expected equity return relates positively to asset volatility and those whose expected equity return relates negatively to asset volatility in portfolio ‘5b’, then my findings again support proposition [2].

In Panel B, I repeat the OLS regressions using equity returns adjusted for the Fama and French (1993) and Carhart (1997) risk factors. In this case, the equity returns of all portfolios are no longer significantly related to the difference between the debt re-payment and the expected asset payment, suggesting that the risk factors can capture pricing effects induced through default risk driven by this factor. However, the equity returns of the high default risk portfolios (with again the exception of portfolio ‘5b’) are still often significantly associated with asset volatility, although at an often lower confidence level. Hence, it seems that the risk factors can only partially capture default risk effects driven by asset volatility.

Finally, Table 3 offers the outcomes from OLS regressions of the default risk premium onto 2-month lagged macroeconomic instruments. The style of the table is similar to that of the former table. In Panel A, I choose the equity return spread between quintile portfolios ‘5’ and ‘1’ as proxy for the default risk premium. One disadvantage of this proxy is that, although the average default probability of portfolio ‘1’ is always close to zero, the one of portfolio ‘5’ can vary substantially over time. As a result, I also approximate the default risk premium through the equity return spread between portfolio ‘5b’ and ‘1’. At least on the portfolio formation date, the average default probability of portfolio ‘5b’ must be strictly greater than 50%. I use the S&P 500 dividend yield (dividend yield), the spread between 10-year and 1-year U.S. T-Bond yields (term spread) and

6The loading on the difference between the debt re-payment and the expected asset payment is now only significant at the 90% confidence level.
the 3-month U.S. T-Bill yield (risk-free rate) as macroeconomic instruments. Empirical evidence in Campbell (1987), Fama and French (1988), Ferson and Harvey (1999), etc. shows that these variables capture the state of the macroeconomy. Moreover, prior studies often suggest that the Fama and French (1993) benchmark risk factors capture a default risk premium (e.g., Fama and French, 1995; Vassalou and Xing, 2004). To test this conjecture, I also regress the equity returns of either HML or SMB on the former more direct empirical proxies for the default risk premium and the macroeconomic instruments (Panel B).

Panel A reveals that both default risk premium proxies are significantly and positively related to the S&P 500 dividend yield and negatively and significantly to the risk-free rate. The S&P 500 dividend yield is approximately equal to the expected market return minus growth (e.g., Fama and French, 1988). As the expected market return should be higher and the risk-free rate of return lower in an economic recession, my empirical findings support the hypothesis that the default risk premium correlates negatively with the state of the economy. Also consistent with intuition, the adjusted $R^2$ derived from the proxy variable based on portfolios ‘5b’ and ‘1’ yields a higher adjusted $R^2$ than that based on portfolio ‘5’ and ‘1’ (1.62% compared to 1.05%, respectively). My empirical evidence in Panel B shows that only the default risk premium based on portfolios ‘5b’ and ‘1’ helps to forecast SMB at the 90% confidence level. Still, the relation between the default risk premium proxy and SMB is negative, indicating that SMB performs well when the default risk premium is low. Overall, the associations between the benchmark risk factors and my default risk premium proxies are weak at best.

4. Conclusions

In this study, I explore the relation between default risk and the expected equity return within the classical framework of Rubinstein (1976). My theoretical findings illustrate that changes in default risk induced through the expected asset payment and the debt re-payment (or both) always positively correlate with changes in the expected equity return. However, changes in default risk induced through asset volatility can both increase or decrease the expected equity return. The reason is as follows: while an increase in the survival probability increases the chance that equityholders obtain a positive payoff, it can also shift equity payoff probability mass from low to high equity payoffs. The shift of probability mass strongly increases the expected equity payment, but it has a weaker positive effect on the equity price. The weaker effect on the equity price stems from the fact that risk-averse equity investors value low equity payoffs on average more positively than high equity payoffs. Depending on the strength of the two effects and thus the equity price increase, the expected equity return can both increase or decrease. I also show that the default risk premium, i.e., the spread in expected equity returns between a high and a low default risk firm, depends on macroeconomic conditions. Preliminary empirical evidence is consistent with the implications of the model.
References


A Appendix: Proofs

Let $H[x]$ be defined as $n[x]/N[-x]$, which can be interpreted as the hazard function for the normally-distributed random variable $x$. The following lemma will be useful throughout the subsequent proofs:

**Lemma A.1:** The hazard function $H[x]$ exhibits the properties that (i) $H[x] > x$, (ii) $0 < H'[x] < 1$ and (iii) $H''[x] > 0$, where $H'[x]$ and $H''[x]$ denote the first derivative and the second derivative of the hazard function with respect to its input argument $x$, respectively.

**Proof:** See Chechile (2003) and Freeman and Guermat (2006).

**Proof of Lemma 1:**

If $\sigma_x > B\kappa \hat{\sigma}_C$, the partial derivative is positive, as $pe^{\mu_C + \frac{1}{2} \sigma_x^2} > 0$ and as both terms in the sum in square parentheses in equation [5] are positive. If $\sigma_x < B\kappa \hat{\sigma}_C$, the partial derivative has the same sign as the term in square parentheses. This term will be positive (negative), if and only if:

$$-(\sigma_x - B\kappa \hat{\sigma}_C)e^{n/2[\sigma_x^2 + 2\sigma_xC]}N\left[\frac{\mu_x + \sigma_xC + \sigma_x^2 - \ln k}{\sigma_x}\right] < (>) n\left[\frac{\mu_x + \sigma_xC + \sigma_x^2 - \ln k}{\sigma_x}\right].$$

Rearranging:

$$-(\sigma_x - B\kappa \hat{\sigma}_C) < (>) e^{-(\mu_x - \ln k + \frac{1}{2}[\sigma_x^2 + 2\sigma_xC])}n\left[\frac{\mu_x + \sigma_xC + \sigma_x^2 - \ln k}{\sigma_x}\right] / N\left[\frac{\mu_x + \sigma_xC + \sigma_x^2 - \ln k}{\sigma_x}\right].$$

As $H[x] > x$ and $H'[x] > 0$, it is obvious that inequality [13] will hold with a “smaller than”-sign, if $(\mu_x - \ln k)/\sigma_x \leq 0$, in which case the partial derivative of the equity price with respect to volatility would be positive. However, as $\lim_{x \to -\infty} H[x] = 0$, the term on the right-hand side of inequality [13] can be rendered arbitrarily close to zero by making $(\mu_x - \ln k)/\sigma_x$ sufficiently positive. As a result, there must be a strictly positive threshold level $\theta^*$, for which inequality [13] will hold with a “greater than”-sign, if and only if $(\mu_x - \ln k)/\sigma_x > \theta^*$. In this case, the partial derivative of the equity price with respect to volatility would be negative.

In summary, $\sigma_x > Bn\hat{\sigma}_C$ or $\sigma_x < Bn\hat{\sigma}_C$ and $(\mu_x - \ln k)/\sigma_x < \theta^*$ are necessary and sufficient conditions for a positive sign of the partial derivative of the equity price with respect to volatility. In contrast, $\sigma_x < B\kappa \hat{\sigma}_C$ and $(\mu_x - \ln k)/\sigma_x > \theta^*$ are necessary and sufficient conditions for a negative sign of the partial derivative.
Proof of Proposition 1:

(i) and (ii): In the main text, it is shown that the expected asset payment ($\mu_x$) and default risk are negatively related, while debt ($k$) and default risk are positively related. If the expected equity return ($E[R_E]$) decreases in the expected asset payment and increases in debt, then an increase in default risk induced through either one of these two factors would also lead to an increase in the expected equity return, and vice versa. Conversely, if the expected equity return increases in the expected asset payment and decreases in debt, then a change in default risk induced through either one of these two factors would lead to an opposite change in the expected equity return. To analyze the relations of the expected equity return with the expected asset payment and debt, I first take the partial derivative of the expected equity return with respect to $\mu_x$:

$$\frac{\partial E[R_E]}{\partial \mu_x} = \frac{1}{p_E^2} \left[ e^{\mu_x + \frac{1}{2} \sigma_x^2} N \left( \frac{\mu_x + \sigma_x^2 - \ln k}{\sigma_x} \right) \left( \frac{\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{\sigma_x} \right) \right. $$

$$N \left[ \frac{\mu_x + \sigma_x^2 + \sigma_C^2 - \ln k}{\sigma_x} \right] - \frac{\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{\sigma_x} N \left[ \frac{\mu_x + \sigma_x^2 + \sigma_C^2 - \ln k}{\sigma_x} \right] - kN \left[ \frac{\mu_x - \ln k}{\sigma_x} \right] $$

$$= \frac{-\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{p_E^2} \left[ N \left[ \frac{\mu_x + \sigma_x^2 - \ln k}{\sigma_x} \right] - \frac{\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{\sigma_x} N \left[ \frac{\mu_x + \sigma_x^2 + \sigma_C^2 - \ln k}{\sigma_x} \right] \right] $$

$$- \sigma_{x,C} N \left[ \frac{\mu_x - \ln k}{\sigma_x} \right] \frac{\sigma_x^2}{\sigma_x^2} \left[ k \left[ \frac{\mu_x + \sigma_x^2 - \ln k}{\sigma_x} \right] - \frac{\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{\sigma_x} N \left[ \frac{\mu_x + \sigma_x^2 + \sigma_C^2 - \ln k}{\sigma_x} \right] \right] $$

$$= \frac{-\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{p_E^2} \left[ N \left[ \frac{\mu_x + \sigma_x^2 - \ln k}{\sigma_x} \right] \frac{\sigma_x^2}{\sigma_x^2} - \frac{\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{\sigma_x} N \left[ \frac{\mu_x + \sigma_x^2 + \sigma_C^2 - \ln k}{\sigma_x} \right] \right] $$

$$= \frac{-\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{p_E} \left[ \frac{\sigma_x^2}{\sigma_x^2} - \frac{\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{\sigma_x} N \left[ \frac{\mu_x + \sigma_x^2 + \sigma_C^2 - \ln k}{\sigma_x} \right] \right] $$

$$= -\frac{1}{k} \frac{\partial E[R_E]}{\partial \mu_x}. \quad (10)$$

The partial derivative of the expected equity return with respect to $k$ equals:

$$\frac{\partial E[R_E]}{\partial k} = \frac{1}{p_E^2} \left[ -N \left[ \frac{\mu_x - \ln k}{\sigma_x} \right] \frac{\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{\sigma_x} N \left[ \frac{\mu_x + \sigma_x^2 + \sigma_C^2 - \ln k}{\sigma_x} \right] \right] $$

$$- \sigma_{x,C} N \left[ \frac{\mu_x - \ln k}{\sigma_x} \right] \frac{\sigma_x^2}{\sigma_x^2} \left[ -kN \left[ \frac{\mu_x - \ln k}{\sigma_x} \right] \right]$$

$$= \frac{-\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{p_E} \left[ \frac{\sigma_x^2}{\sigma_x^2} - \frac{\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_{x,C} + \sigma_C^2)}}{\sigma_x} N \left[ \frac{\mu_x + \sigma_x^2 + \sigma_C^2 - \ln k}{\sigma_x} \right] \right] $$

$$= \frac{-1}{k} \frac{\partial E[R_E]}{\partial \mu_x}. \quad (11)$$

which shows that the two partial derivatives must have opposite signs.

As the time preference parameter ($\rho$), debt ($k$), and the exponential function are strictly positive, the coefficients on the term in square parentheses in equations [10] and [11] are negative and positive, respectively. As a result, the signs of the two partial derivatives depend on the sign of the term in square parenthesis. To find
a positive relation between default risk and the expected equity return, the term in square parenthesis must be positive, which will hold if and only if:

\[ N \left[ \frac{\mu_x + \sigma_x^2 - \ln k}{\sigma_x} \right] N \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right] > e^{\sigma_{x,C}} N \left[ \frac{\mu_x - \ln k}{\sigma_x} \right] N \left[ \frac{\mu_x + \sigma_x^2 + \sigma_{x,C} - \ln k}{\sigma_x} \right]. \]

As \( N[\cdot] > 0 \), the last inequality is equivalent to:

\[ \frac{N \left[ \frac{\mu_x + \sigma_x^2 - \ln k}{\sigma_x} \right]}{N \left[ \frac{\mu_x - \ln k}{\sigma_x} \right]} > e^{\sigma_{x,C}} \frac{N \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]}{N \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]} \] (12)

If \( \kappa = 0 \), then \( \sigma_{x,C} = -B\kappa \tilde{\sigma}_x \tilde{\sigma}_C = 0 \) and therefore:

\[ \frac{N \left[ \frac{\mu_x + \sigma_x^2 - \ln k}{\sigma_x} \right]}{N \left[ \frac{\mu_x - \ln k}{\sigma_x} \right]} = e^{\sigma_{x,C}} \frac{N \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]}{N \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]} \] \( \kappa = 0 \).

Only the term on the right-hand side of the former equality features the correlation coefficient \( \kappa \). If this term were monotonically related to \( \kappa \), then we could make inequality [12] hold for either \( \kappa \) greater than zero or \( \kappa \) smaller than zero. Notice that the partial derivative of this term with respect to \( \kappa \) has the same sign as the partial derivative of the natural logarithm of this term with respect to \( \kappa \). Taking the natural logarithm of the term on the right-hand side, we obtain:

\[ \sigma_{x,C} + \ln \left( N \left[ \frac{\mu_x + \sigma_x + \sigma_{x,C} - \ln k}{\sigma_x} \right] \right) - \ln \left( N \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right] \right). \]

Taking the derivative of the last term with respect to \( \kappa \) and rearranging:

\[-B\tilde{\sigma}_x \tilde{\sigma}_C \left[ 1 - \left( \frac{n \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]}{N \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]} - \frac{n \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]}{N \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]} \right) \right] \]

\[-B\tilde{\sigma}_x \tilde{\sigma}_C \left[ 1 - \left( \frac{n \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]}{N \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]} - \frac{n \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]}{N \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]} \right) \right] \]

\[-B\tilde{\sigma}_x \tilde{\sigma}_C \left[ 1 - \left( \frac{H \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right] + \sigma_x}{\tilde{\sigma}_x} - H \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right] \right) \right]. \]

Using the mean-value theorem (e.g., see Simon and Blume, 1994):

\[-B\tilde{\sigma}_x \tilde{\sigma}_C \left( 1 - H'[x^*] \right) , \]

28
where \( x^* \in [-(\mu_x + \sigma_x^2 + \sigma_xC - \ln k)/\sigma_x, -(\mu_x + \sigma_x^2 + \sigma_xC - \ln k)/\sigma_x] \). As \( H' [x] < 1 \), the partial derivative must be negative. In conclusion, if \( \kappa \) is greater than zero, then inequality [12] will hold, and therefore \( \frac{\partial E[R_E]}{\partial \mu_x} < 0 \) and \( \frac{\partial E[R_E]}{\partial k} > 0 \). In other words, if \( \kappa > 0 \), then an increase in default risk induced through either \( \mu_x \) or \( k \) leads to an increase in the expected equity return. Conversely, it is easy to see that, if \( \kappa < 0 \), then an increase in default risk induced through one of these two factors decreases the expected equity return.

(iii) We now examine the impact of a simultaneous change in \( \mu_x \) and \( k \) on the expected equity return. As the cumulative normal density function is a monotonic transformation of its input argument, a change in default risk, i.e., \( N[(\ln k - \mu_x)/\sigma_x] \), must have the same sign as \( (\ln k - \mu_x)/\sigma_x \). For notational convenience, let us define \( \psi(\mu_x, k) \equiv (\ln k - \mu_x)/\sigma_x \). We can write the total derivative of \( \psi(\mu_x, k) \) as:

\[
d\psi(\mu_x, k) = \frac{\partial \psi(\mu_x, k)}{\partial \mu_x} d\mu_x + \frac{\partial \psi(\mu_x, k)}{\partial k} dk = -\frac{d\mu_x}{\sigma_x} + \frac{dk}{k\sigma_x}.
\]

As a result, if \( d\mu_x > (<) \frac{dk}{k} \), then a simultaneous change in \( \mu_x \) and \( k \) leads to a decrease (increase) in default risk. The total derivative of the expected equity return can be written as:

\[
dE[R_E] = \frac{\partial E[R_E]}{\partial \mu_x} d\mu_x + \frac{\partial E[R_E]}{\partial k} dk.
\]

Using equation [11]:

\[
dE[R_E] = \frac{\partial E[R_E]}{\partial \mu_x} d\mu_x - \frac{1}{k} \frac{\partial E[R_E]}{\partial \mu_x} dk = \frac{\partial E[R_E]}{\partial \mu_x} (d\mu_x - \frac{dk}{k}).
\]

Recall that, if \( \kappa > 0 \), \( \frac{\partial E[R_E]}{\partial \mu_x} < 0 \). As an increase (decrease) in default risk implies \( d\mu_x < (> \) \frac{dk}{k} \), the expected equity return increases (decreases) with increases (decreases) in default risk induced through both \( \mu_x \) and \( k \). Conversely, if \( \kappa < 0 \), one can easily show that the expected equity return decreases (increases) with increases (decreases) in default risk induced through both factors.

**Proof of Corollary 1:**

We now analyze the dependence of the association between default risk and the expected equity return on the firm’s absolute level of default risk. As \( \lim N[\psi(\mu_x, k)]|_{\mu_x \to -\infty} = 0 \) and \( \lim N[\psi(\mu_x, k)]|_{k \to 0} = 0 \), we can decrease a firm’s default risk to zero by either increasing its expected asset payment to infinity or by decreasing its debt to zero. In the limit, the relation between \( E[R_E] \) and \( \mu_x \) will be:

\[
\lim_{\mu_x \to -\infty} \frac{\partial E[R_E]}{\partial \mu_x} = \lim_{\mu_x \to -\infty} -\frac{\rho k e^{\mu_x} + \mu_C + \frac{1}{2} (\sigma_x^2 + \sigma_x^2)}{p^2} \left[ N \left( \frac{\mu_x + \sigma_x^2 - \ln k}{\sigma_x} \right) N \left( \frac{\mu_x + \sigma_xC - \ln k}{\sigma_x} \right) - e^{-\sigma_x C} N \left( \frac{\mu_x - \ln k}{\sigma_x} \right) N \left( \frac{\mu_x + \sigma_x^2 + \sigma_xC - \ln k}{\sigma_x} \right) \right].
\]
Using the closed-form solution of \( p_E \), we can write the first term in the product as:

\[
\frac{\rho k e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + \sigma_C^2)}}{\left(\rho e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + \sigma_C^2)}\right) e^{\sigma_x, C} N \left[ \frac{\mu_x + \sigma_x, C + \frac{1}{2}(\sigma_x^2 + \sigma_C^2)}{\sigma_x} - k e^{\mu_x + \frac{1}{2}(\sigma_x^2 + \sigma_C^2)} N \left[ \frac{\mu_x + \sigma_x, C - ln k}{\sigma_x} \right] \right]}
\]

As \( \mu_x \to \infty \), the cumulative normal density functions converge to unity and \( e^{-\frac{1}{2}\sigma_x^2} \) converges to zero. As a result, the squared term converges to \( e^{2\sigma_x, C} \), which is a strictly positive number. In contrast, \( e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + \sigma_C^2)} \) converges to infinity. By the product rule of limits, the denominator then converges to infinity, which implies that the first term in the product converges to zero.

Regarding the second term in the product, we once again note that the cumulative density functions converge to unity, as \( \mu_x \) converges to infinity. As a result, the second term converges to \( 1 - e^{\sigma_x, C} \). If we rule out that \( e^{\sigma_x, C} = 1 \), then, again by the product rule of limits, the limit of the partial derivative of the expected equity return with respect to the expected asset payment is equal to zero. As \( \frac{\partial E[R_E]}{\partial k} = -\frac{1}{\rho} \frac{\partial E[R_E]}{\partial \mu_x} \), the limit of \( \frac{\partial E[R_E]}{\partial k} \) is also zero. In other words, the expected equity returns of firms whose default risk is very close to zero should not be affected by changes in the expected asset payment and/or debt.

**Proof of Proposition 2:**

In the main text, I have shown that the relation between volatility and the expected equity return depends on a firm’s level of default risk. If default risk is below (above) 50%, then it is positively (negatively) related to changes in asset volatility. To find a positive relation between default risk induced through asset volatility and the expected equity return, the expected equity return must increase in volatility, if default risk is below 50%, and decrease in volatility, if default risk is above 50%. The relation between volatility and the expected equity return can be determined by the following partial derivative:

\[
\frac{\partial E[R_E]}{\partial \sigma_x} = \frac{1}{p_E^2} \left( \sigma_x e^{\mu_x + \frac{1}{2}\sigma_x^2} N \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - ln k \right] + k N \left[ \frac{\mu_x - ln k}{\sigma_x} \right] \right) \left( e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_x, C + \sigma_C^2)} \right)
\]

\[
\frac{\sigma_x e^{\mu_x + \frac{1}{2}\sigma_x^2} N \left[ \frac{\mu_x + \sigma_x, C + \frac{1}{2}(\sigma_x^2 + \sigma_C^2)}{\sigma_x} - k e^{\mu_x + \frac{1}{2}(\sigma_x^2 + \sigma_C^2)} N \left[ \frac{\mu_x + \sigma_x, C - ln k}{\sigma_x} \right] \right] - \rho k e^{\mu_x + \frac{1}{2}(\sigma_x^2 + \sigma_C^2)} N \left[ \frac{\mu_x + \sigma_x, C - ln k}{\sigma_x} \right] - \left( e^{\mu_x + \frac{1}{2}\sigma_x^2} \right)
\]

\[
N \left[ \frac{\mu_x + \sigma_x, C + \frac{1}{2}(\sigma_x^2 + \sigma_C^2)}{\sigma_x} - k N \left[ \frac{\mu_x - ln k}{\sigma_x} \right] \right) \left( e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_x, C + \sigma_C^2)} \right)
\]

\[
N \left[ \frac{\mu_x + \sigma_x, C + \frac{1}{2}(\sigma_x^2 + \sigma_C^2)}{\sigma_x} - ln k \right) + k e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + 2\sigma_x, C + \sigma_C^2)} N \left[ \frac{\mu_x + \sigma_x, C - ln k}{\sigma_x} \right] \right)
\].
This partial derivative is positive, if and only if:

\[
\sigma_x e^{\mu_x + \frac{1}{2} \sigma_x^2} N \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right] + k \left[ \frac{\mu_x - \ln k}{\sigma_x} \right] \left( \rho e^{\mu_C + \frac{1}{2} (\sigma_C^2 + 2 \sigma_x \sigma_C + \sigma_C^2)} - \frac{k \rho e^{\mu_C + \frac{1}{2} (\sigma_C^2 + 2 \sigma_x \sigma_C + \sigma_C^2)}}{\sigma_x} \right) \\left( \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right) > \left( e^{\mu_x + \frac{1}{2} \sigma_x^2} \right) \left( \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right)
\]

As \( e^{\mu_x + \frac{1}{2} \sigma_x^2} N \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right] - k N \left[ \frac{\mu_x - \ln k}{\sigma_x} \right] = E[\max(\tilde{x} - k, 0)] > 0 \) and as \( e^{\mu_x + \frac{1}{2} \sigma_x^2} N \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right] \), the former inequality is equivalent to:

\[
\frac{\sigma_x e^{\mu_x + \frac{1}{2} \sigma_x^2} N \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right] + k \left[ \frac{\mu_x - \ln k}{\sigma_x} \right]}{e^{\mu_x + \frac{1}{2} \sigma_x^2} N \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right] - k N \left[ \frac{\mu_x - \ln k}{\sigma_x} \right]}
\]

\[
= \frac{(\sigma_x - B \tilde{\sigma} C) e^{\mu_x + \frac{1}{2} \sigma_x^2 + \sigma_C} N \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right] + k \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right]}{(\sigma_x - B \tilde{\sigma} C) + \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right] N \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right]}
\]

\[
= \frac{(\sigma_x - B \tilde{\sigma} C)}{1 - \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right] / \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right]}
\]

(13)

If \( \kappa = 0 \), then \( \sigma_x = 0 \). As a result:

\[
\frac{\sigma_x e^{\mu_x + \frac{1}{2} \sigma_x^2} N \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right] + k \left[ \frac{\mu_x - \ln k}{\sigma_x} \right]}{e^{\mu_x + \frac{1}{2} \sigma_x^2} N \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right] - k N \left[ \frac{\mu_x - \ln k}{\sigma_x} \right]} = \frac{(\sigma_x - B \tilde{\sigma} C) + \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right]}{1 - \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right] / \left[ \frac{\mu_x + \sigma_x^2}{\sigma_x} - \ln k \right]}
\]

Once again, we notice that only the term on the right-hand side of the former equality features \( \kappa \). If this term were monotonically related to \( \kappa \), then inequality [13] would hold for either \( \kappa > 0 \) or \( \kappa < 0 \). For notational convenience, let us define \( \alpha \equiv (\ln k - \mu_x) / \sigma_x, \alpha^* \equiv \sigma_x > 0 \) and \( \beta \equiv B \tilde{\sigma} C > 0 \). Using these definitions, we can
write the term on the right-hand side as:

\[
\frac{\alpha^* - \beta \kappa + H [\alpha - \alpha^* + \beta \kappa]}{1 - H [\alpha - \alpha^* + \beta \kappa] / H [\alpha + \beta \kappa]}.
\]

The partial derivative of this term with respect to \( \kappa \) has the same sign as:

\[
\beta \left[ H' [\alpha - \alpha^* + \beta \kappa] - 1 \right] \left[ H [\alpha + \beta \kappa] - H [\alpha - \alpha^* + \beta \kappa] + H [\alpha - \alpha^* + \beta \kappa] \right] + \beta \left[ \alpha^* - \beta \kappa + H [\alpha - \alpha^* + \beta \kappa] \right] \cdot \frac{H' [\alpha - \alpha^* + \beta \kappa] / H [\alpha + \beta \kappa] - H [\alpha - \alpha^* + \beta \kappa] H' [\alpha + \beta \kappa] / H [\alpha + \beta \kappa]^2}{(1 - H [\alpha - \alpha^* + \beta \kappa] / H [\alpha + \beta \kappa])}
\]

Multiplying by \( 1 / \beta > 0 \) and \( H [\alpha + \beta \kappa] > 0 \), adding and subtracting \( \alpha \) inside the third main term, using the relation that \( H' [x] = H [x] [H [x] - x] \), and rearranging yields:

\[
\left[ H' [\alpha - \alpha^* + \beta \kappa] - 1 \right] H' [c^*] + H' [\alpha - \alpha^* + \beta \kappa] [1 - H' [c^*]] + \alpha H [\alpha - \alpha^* + \beta \kappa] [1 - H' [c^*]]
\]

\[
= -H' [c^*] + H' [\alpha - \alpha^* + \beta \kappa] + \alpha H [\alpha - \alpha^* + \beta \kappa] [1 - H' [c^*]]
\]

where \( c^* \in [\alpha + \beta \kappa, \alpha - \alpha^* + \beta \kappa] \).

We should note that \( c^* > \alpha - \alpha^* + \beta \kappa \). As \( H [x] \) is a convex function, the sum of the first two terms in formula [16] is negative. As \( H [x] \) is positive and \( H' [x] < 1 \), if a firm’s default probability is below 50%, i.e., \( \alpha = (\ln k - \mu_\gamma) / \sigma_\gamma < 0 \), or equal to 50%, i.e., \( \alpha = 0 \), the third term is negative or zero, respectively. In total, the sign of the partial derivative of the term on the right-hand side with respect to \( \kappa \) is negative, if a firm’s default risk is smaller or equal to 50%. In turn, this implies that, conditional on \( \kappa > 0 \) and default risk \( \leq 50\% \), inequality [13] holds and the expected equity return increases in asset volatility. As default risk increases in asset volatility for absolute levels of default risk below 50%, there exists a positive relation between default risk induced through asset volatility and the expected equity return, if default risk \( < 50\% \). However, if default risk \( = 50\% \), then asset volatility does not affect default risk, while the expected equity return still increases in asset volatility.

If default risk is above 50%, i.e., if \( \alpha = (\ln k - \mu_\gamma) / \sigma_\gamma > 0 \), then the scaled partial derivative in formula [16]

\[
\frac{\alpha^* - \beta \kappa + H [\alpha - \alpha^* + \beta \kappa]}{1 - H [\alpha - \alpha^* + \beta \kappa] / H [\alpha + \beta \kappa]}
\]

\[\text{The partial derivative is the term in formula [14] divided by } (1 - H [\alpha - \alpha^* + \beta \kappa] / H [\alpha + \beta \kappa])^2.\]
can be positive or negative. To see that it can be negative, note that, at \( \alpha = 0 \), the scaled partial derivative is strictly negative. By continuity, the scaled partial derivative must then also be negative for values of \( \alpha \) slightly greater than zero. To see that it can be positive, reconsider the scaled partial derivative written as in formula [15]. As \( \lim_{x \to -\infty} H'[x] = 1 \), \( \lim_{x \to -\infty} H[x] = x \), and \( \lim_{x \to -\infty} H[x] = \infty \), it is obvious that this term converges to positive infinity, as \( \alpha \to \infty \). The partial derivative must then also converge to positive infinity. In other words, if a firm’s default risk is above 50\%, then its expected equity return can both increase or decrease in its asset volatility, while its default risk can only decrease in its asset volatility.

To shed some more light on the relation between asset volatility and the expected equity return, I plot one specific example of the scaled partial derivative in formula [14] in Figure A.1. In the figure, I choose \( B, \sigma_x, \hat{\sigma}_C \) and \( \kappa \) to be 0.001, 0.65, 0.15 and 0.45, respectively. As a direct consequence, \( \alpha^* \) equals 0.65 and \( \beta \) equals 0.00025. I choose the values of \( \alpha \) such that default risk increases from 0.5\% to 99.5\% in increments of 0.5\%. As can be seen, the value of the scaled partial derivative is always negative for firms with a default probability below 50\%, implying a positive association between the expected equity return and asset volatility. As the default risk of these firms also increases in asset volatility, their expected equity return associates positively with default risk. In contrast, the scaled partial derivative can be positive or negative for firms with a default probability above 50\%. In fact, when the default probability is only slightly greater than 50\%, the scaled partial derivative is negative. On the other hand, if the default probability is much greater than 50\%, then the scaled partial derivative is positive. As a result, the expected equity return of these firms can increase, remain constant or decrease in asset volatility, while their default risk always decreases in asset volatility. In this case, there is no monotone relation between the expected equity return and default risk.

**Proof of Proposition 3:**

Recall that, for a constant asset volatility, the total derivative of the expected equity return with respect to changes in both the expected asset payment and debt equals:

\[
\frac{dE[R_E]}{d\mu_x} = \frac{\partial E[R_E]}{\partial \mu_x} \, d\mu_x - \frac{1}{k} \frac{\partial E[R_E]}{\partial \mu_x} \, dk = \frac{\partial E[R_E]}{\partial \mu_x} \left( d\mu_x - \frac{dk}{k} \right),
\]

with:

\[
\frac{\partial E[R_E]}{\partial \mu_x} = -\rho k e^{\mu_x + \mu_C + \frac{1}{2}(\sigma_x^2 + \sigma_C^2)} \left[ N \left( \frac{\mu_x + \sigma_x^2 - \ln k}{\sigma_x} \right) \left( \frac{\mu_x + \sigma_x^2 + \sigma_x C - \ln k}{\sigma_x} \right) \right] - e^{\sigma_x C} N \left( \frac{\mu_x - \ln k}{\sigma_x} \right) \left( \frac{\mu_x + \sigma_x^2 + \sigma_x C - \ln k}{\sigma_x} \right) .
\]

(i) and (ii): Note that only the first term in the product is related to the time preference parameter (\( \rho \)) and
consumption growth \((\mu_C)\). As in the proof of corollary 1, we can write this first term as:

\[
\frac{\rho e^{\mu_x + \mu_C + \frac{1}{2} (\sigma_x^2 + \sigma_C^2)}}{p^E} = -\frac{k}{\rho e^{\mu_x + \mu_C + \frac{1}{2} (\sigma_x^2 + \sigma_C^2)} \left( e^{\sigma_{x,C} N \left( \frac{\mu_x + \sigma_{x,C} + \sigma_C^2 - \ln k}{\sigma_x} \right)} - k e^{-\frac{1}{2} \sigma^2 N \left[ \frac{\mu_x + \sigma_{x,C} - \ln k}{\sigma_x} \right]} \right)^2},
\]

As \(\mu_C = C_0 - B \hat{\mu}_C\), an increase in expected future consumption \((\hat{\mu}_C)\) decreases the denominator of this term, and therefore makes the whole term more negative. In contrast, an increase in consumption today \((C_0)\) or the time preference parameter \((\rho)\) increases the denominator of this term, and in turn makes the whole term less negative. As a result, the magnitude of the relation between default risk simultaneously induced through the expected asset payment and debt and the expected equity return increases in expected future consumption and decreases in consumption today and the time preference parameter.

(iii): The proof of proposition 2 shows that the term in square parenthesis in formula [17] increases from \(\kappa = 0\) to any positive value of \(\kappa\). It is now sufficient to show that the equity price decreases in \(\kappa\), which can easily be done by taking the partial derivative of the equity price with respect to \(\kappa\).
Figure A1: The figure plots the value of the scaled partial derivative in formula [14] against default probabilities ranging from 0.5% to 99.5%, i.e., the value of $[(\ln k - \mu_x)/\sigma_x]$ is chosen to be equal to $N^{-1}[d_j]$. I assume that $B$ is 0.001, $\sigma_x$ is 0.65, $\sigma_C$ is 0.15 and $\kappa$ is 0.45. The solid vertical line denotes a 50% default probability, while the broken horizontal line denotes a zero value of the scaled partial derivative.
B Appendix: The Politis and Romano (1994) Bootstrap

I now shortly summarize my implementation of the Politis and Romano (1994) bootstrap. Some advantages of this procedure are that it can take account of both autocorrelation and heteroscedasticity in the error term, and that it considers dependence between the exogenous variables and the error term.

1. Perform a whole-sample OLS regression of $Y$ on $X$ and store the estimates.

2. Draw a random integer $r$ from a uniform distribution on $[1,T]$, where $T$ is the total number of whole sample observations. Store $X_r$ as $X_{bs,t=1}$, where $X_r$ contains the $r^{th}$ observations of the whole sample exogenous variables and $X_{bs,t=1}$ contains the first observations of the bootstrap sample exogenous variables. Imposing the null, use $X_r$, $\varepsilon_r$ and the whole-sample OLS estimates to construct the bootstrap dependent variable $Y_{bs,t=1}$, where $\varepsilon_r$ is the $r^{th}$ observation of the residual from the whole-sample OLS regression. Draw a continuous uniform random variable $p$ on $[0,1]$. If $p > 0.5$, use the next observations from the whole sample versions of $X$ and $\varepsilon$ to augment the bootstrap sample, i.e., to construct $Y_{bs,t=2}$ and $X_{bs,t=2}$. Else, draw a new random integer $r$ from the uniform distribution on $[1,T]$ to augment the bootstrap sample. Continue until $Y_{bs}$ and $X_{bs}$ have the same number of observations as $Y$ and $X$.

3. Run an OLS regression using the bootstrap sample and store the t-statistic of the parameter on which the null was imposed.

4. Repeat steps (2) and (3) 1,000 times.

5. Sort the resulting 1,000 t-statistics in ascending order. Use the 25$^{th}$ observation as the 2.5% critical value and the 975$^{th}$ observation as the 97.5% critical value.

As the exogenous variables used in this study show a high degree of dependence over time, I modify the original procedure by not sampling over the exogenous variables (see Goyal and Welsh, 2007).
Table 1: Average Equity Return and Merton (1974) Default Risk

<table>
<thead>
<tr>
<th>Default Risk Quintiles</th>
<th>No Debt</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>5a (dr&lt;50%)</th>
<th>5b (dr&gt;50%)</th>
<th>5-1</th>
<th>5b-5a</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Raw returns</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># stocks</td>
<td>542</td>
<td>674</td>
<td>671</td>
<td>668</td>
<td>666</td>
<td>632</td>
<td>463</td>
<td>169</td>
<td></td>
<td></td>
</tr>
<tr>
<td>return</td>
<td>1.02***</td>
<td>0.98***</td>
<td>1.00***</td>
<td>1.07***</td>
<td>1.06***</td>
<td>1.12***</td>
<td>1.07***</td>
<td>1.23***</td>
<td>0.14</td>
<td>0.16</td>
</tr>
<tr>
<td>default risk</td>
<td>0.03***</td>
<td>0.07**</td>
<td>0.33***</td>
<td>0.55***</td>
<td>2.54***</td>
<td>16.99***</td>
<td>12.25***</td>
<td>44.18***</td>
<td>16.91***</td>
<td>31.93***</td>
</tr>
<tr>
<td>ln(Debt) - E[Asset Payment]</td>
<td>-4.24***</td>
<td>-3.57***</td>
<td>-2.11***</td>
<td>-1.63***</td>
<td>-1.28***</td>
<td>-0.72***</td>
<td>-0.87***</td>
<td>0.23***</td>
<td>2.84***</td>
<td>1.10***</td>
</tr>
<tr>
<td>Volatility</td>
<td>0.41***</td>
<td>0.26***</td>
<td>0.29***</td>
<td>0.30***</td>
<td>0.36***</td>
<td>0.51***</td>
<td>0.46***</td>
<td>0.81***</td>
<td>0.25***</td>
<td>0.35***</td>
</tr>
<tr>
<td><strong>Panel B: Orthogonalized returns (with respect to FF/C risk factors)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># stocks</td>
<td>377</td>
<td>512</td>
<td>509</td>
<td>507</td>
<td>506</td>
<td>482</td>
<td>373</td>
<td>110</td>
<td></td>
<td></td>
</tr>
<tr>
<td>return</td>
<td>0.67***</td>
<td>0.61***</td>
<td>0.52***</td>
<td>0.50***</td>
<td>0.46***</td>
<td>0.54***</td>
<td>0.46***</td>
<td>0.94***</td>
<td>-0.07</td>
<td>0.48</td>
</tr>
<tr>
<td>default risk</td>
<td>0.03***</td>
<td>0.07**</td>
<td>0.26***</td>
<td>0.50***</td>
<td>1.84***</td>
<td>14.72***</td>
<td>10.95***</td>
<td>43.59***</td>
<td>14.65***</td>
<td>32.63***</td>
</tr>
<tr>
<td>ln(Debt) - E[Asset Payment]</td>
<td>-4.16***</td>
<td>-3.60***</td>
<td>-2.13***</td>
<td>-1.65***</td>
<td>-1.29***</td>
<td>-0.77***</td>
<td>-0.88***</td>
<td>0.31***</td>
<td>2.82***</td>
<td>1.19***</td>
</tr>
<tr>
<td>Volatility</td>
<td>0.39***</td>
<td>0.26***</td>
<td>0.28***</td>
<td>0.30***</td>
<td>0.34***</td>
<td>0.47***</td>
<td>0.43***</td>
<td>0.75***</td>
<td>0.21***</td>
<td>0.32***</td>
</tr>
</tbody>
</table>

a In this table, I show time-series averages of several characteristics of portfolios one-way sorted on Merton (1974) default risk. The first characteristic is the number of firms contained in each portfolio (# stocks). The other characteristics, which are value-weighted averages taken over the firms in each portfolio, consist of the equity return (return), the Merton (1974) default probability (default risk), the difference between the natural logarithm of total debt and the expected asset payment (ln(Debt) − E[Asset Payment]) and the asset volatility (volatility). In Panels A and B, I analyze either raw equity returns or equity returns orthogonalized with respect to the benchmark risk factors proposed by Fama and French (1993) and Carhart (1997) (i.e., the market return, SMB, HML and WML). The orthogonalizations are done through 4-year out-of-sample rolling window regressions of equity returns onto the benchmark risk factors. Portfolios are formed in June of each year t, and then held from July of year t to June of year t + 1. The portfolio labeled ‘No Debt’ contains all firms with zero debt on the portfolio formation date. Portfolios 1-5 are formed according to the quintile breakpoints derived from the Merton (1974) default probability on the portfolio formation date. Portfolio 1 contains all firms with a default probability below the first quintile breakpoint, portfolio 2 those firms with a default probability above the first and below the second breakpoint, and so on. Portfolio 5a and 5b split firms in quintile 5 into those with a below (dr < 50%) and those with an above (dr > 50%) 50% default probability, respectively. Portfolios 5-1 and 5b-5a are long on portfolio 5 and 5b and short on portfolio 1 and 5a, respectively. ***, ** and * indicate statistical significance at the 99%, 95% and 90% confidence level, respectively. The sample period ranges from January 1970 to December 2007.
<table>
<thead>
<tr>
<th>Quintile</th>
<th>Raw returns</th>
<th>Orthogonalized returns (with respect to FF/C risk factors)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quintile 1 (low)</td>
<td>1.76</td>
<td>0.21</td>
</tr>
<tr>
<td>Quintile 2</td>
<td>2.80</td>
<td>0.15</td>
</tr>
<tr>
<td>Quintile 3</td>
<td>2.19</td>
<td>0.25</td>
</tr>
<tr>
<td>Quintile 4</td>
<td>2.19</td>
<td>0.19</td>
</tr>
<tr>
<td>Quintile 5; dp &lt; 50%</td>
<td>1.38</td>
<td>0.21</td>
</tr>
<tr>
<td>Quintile 1 (low)</td>
<td>0.45</td>
<td>0.21</td>
</tr>
<tr>
<td>Quintile 2</td>
<td>2.80</td>
<td>0.15</td>
</tr>
<tr>
<td>Quintile 3</td>
<td>2.19</td>
<td>0.25</td>
</tr>
<tr>
<td>Quintile 4</td>
<td>2.19</td>
<td>0.19</td>
</tr>
<tr>
<td>Quintile 5; dp &lt; 50%</td>
<td>1.38</td>
<td>0.21</td>
</tr>
</tbody>
</table>

In this table, I show the outcomes from time-series OLS regressions of default risk portfolio returns onto a constant, the 2-month lagged difference between the natural logarithm of total debt and the expected asset payment (ln(Debt) - E[Asset Payment]) and the 2-month lagged asset volatility (volatility). In Panels A and B, I analyze either raw equity returns or equity returns orthogonalized with respect to the benchmark risk factors proposed by Fama and French (1993) and Carhart (1997) (i.e., the market return, SMB, HML and WML). The orthogonalizations are done through 4-year out-of-sample rolling window regressions of equity returns onto the benchmark risk factors. Default risk portfolios are formed in June of each year, and then held from July of year $t$ to June of year $t+1$. Portfolios 1-5 are formed according to the quintile breakpoints derived from the Merton (1974) model, and those with a below 50% default probability a below the first quintile breakpoint, portfolio 2 those firms with a default probability above the first and the below the second breakpoint, and so on. Portfolios 5a and 5b split firms in quintile 5 into those with a default probability above the first quintile breakpoint and those with above above (dr > 50%) default probability, respectively. The table shows the estimate in bold (adj. $R^2$ of Politis and Romano (1994) in curly parenthesis (upper cv and lower cv, respectively). In addition, it shows the adjusted $R^2$. The sample period ranges from January 1970 to December 2007.
Table 3: Default Risk Spread Portfolios and Macroeconomic Conditions

<table>
<thead>
<tr>
<th>Dependent variable</th>
<th>Constant estimate</th>
<th>Dividend yield estimate</th>
<th>Term spread estimate</th>
<th>risk-free rate estimate</th>
<th>Quintile 5-1 estimate</th>
<th>Quintile 5b-1 estimate</th>
<th>Adj. R²</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>t-stat</td>
<td>t-stat</td>
<td>t-stat</td>
<td>t-stat</td>
<td>t-stat</td>
<td>t-stat</td>
<td></td>
</tr>
<tr>
<td>Quintile 5-1</td>
<td>0.16 [-0.21]</td>
<td>0.85 [2.19]</td>
<td>-0.33 [-1.99]</td>
<td>-4.43 [-2.42]</td>
<td>1.05%</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quintile 5b-1</td>
<td>-0.22 [-0.22]</td>
<td>1.30 [3.44]</td>
<td>-0.41 [-1.84]</td>
<td>-6.13 [-2.55]</td>
<td>1.62%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Panel A: Default risk spread portfolios

Panel B: Fama and French benchmark factors

SMB

<table>
<thead>
<tr>
<th>SMB</th>
<th>-0.10 [-0.19]</th>
<th>0.57 [2.67]</th>
<th>-0.13 [-1.17]</th>
<th>-2.74 [-2.49]</th>
<th>-0.05 [-1.44]</th>
<th>1.38%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[2.20]</td>
<td>[2.01]</td>
<td>[1.96]</td>
<td>[2.12]</td>
<td>[2.18]</td>
<td></td>
</tr>
</tbody>
</table>

SMB

<table>
<thead>
<tr>
<th>SMB</th>
<th>-0.12 [-0.22]</th>
<th>0.60 [2.79]</th>
<th>-0.13 [-1.16]</th>
<th>-2.87 [-2.62]</th>
<th>-0.05 [-1.98]</th>
<th>2.07%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[-1.96]</td>
<td>[-2.04]</td>
<td>[-1.96]</td>
<td>[-2.03]</td>
<td>[-1.99]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[2.10]</td>
<td>[2.06]</td>
<td>[2.17]</td>
<td>[2.02]</td>
<td>[2.18]</td>
<td></td>
</tr>
</tbody>
</table>

HML

<table>
<thead>
<tr>
<th>HML</th>
<th>0.22 [0.32]</th>
<th>-0.28 [-1.09]</th>
<th>0.15 [0.99]</th>
<th>1.81 [1.42]</th>
<th>0.01 [0.50]</th>
<th>-0.28%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[-2.20]</td>
<td>[-2.14]</td>
<td>[-2.05]</td>
<td>[-2.07]</td>
<td>[-2.00]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[2.03]</td>
<td>[2.12]</td>
<td>[2.11]</td>
<td>[2.28]</td>
<td>[1.96]</td>
<td></td>
</tr>
</tbody>
</table>

HML

<table>
<thead>
<tr>
<th>HML</th>
<th>0.22 [0.32]</th>
<th>-0.27 [-1.03]</th>
<th>0.15 [0.98]</th>
<th>1.74 [1.34]</th>
<th>-0.01 [-0.24]</th>
<th>-0.32%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[2.33]</td>
<td>[2.00]</td>
<td>[2.21]</td>
<td>[2.18]</td>
<td>[2.11]</td>
<td></td>
</tr>
</tbody>
</table>

In this table, I show the outcomes from time-series OLS regressions of the returns on default risk spread portfolios (Panel A), SMB and HML (Panel B) on 2-month lagged macroeconomic instruments. The first default risk spread portfolio (Quintile 5-1) is long on firms with a Merton (1974) default probability above the fourth quintile breakpoint and short on firms with a Merton (1974) default probability below the first quintile breakpoint, both on the portfolio formation date. The second default risk quintile (Quintile 5b-1) is defined in an identical way, except that the long position only contains firms with a Merton (1974) default probability both above the fourth quintile breakpoint and above 50% on the portfolio formation date. The macroeconomic instruments are the S&P 500 dividend yield (dividend yield), the spread between the yields of 10-year U.S. government bonds and 3-month U.S. Treasury bills (term spread) and the 3-month risk-free rate (risk-free rate). In Panel B, I add the returns on either one of the two default risk spread portfolios (quintile 5-1 and quintile 5b-1, respectively) as further instruments. The table shows the estimate in bold (estimate), the t-statistic in square parentheses (t-stat), and the upper and lower 95% critical values obtained from the stationary bootstrap procedure of Politis and Romano (1994) in curly parenthesis below the t-statistic. In addition, it shows the adjusted R² (adj. R²). The sample period ranges from January 1970 to December 2007.
Figure 1: The figure shows the impact of a change of the expected asset payment ($\mu_x$) on the probability density function of the asset payment $x$. The value of the probability density function can be read off from the left x-axis. The conditional expectation of the marginal rate of substitution ($E[MRS|x]$) is represented by the convex black line and its value can be read off from the right x-axis. The solid vertical line represents the debt level of a hypothetical firm. The shaded area in the figure represents the integrated difference between the two probability density functions to the right of the debt level.
Figure 2: The figure shows the impact of a change of the asset volatility ($\sigma_x$) on the probability density function of the asset payment $x$. The value of the probability density function can be read off from the left x-axis. The conditional expectation of the marginal rate of substitution ($E[MRS|x]$) is represented by the convex black line and its value can be read off from the right x-axis. The solid vertical line represents the debt level of a hypothetical firm with a default probability slightly above 50%. The shaded area in the figure represents the integrated difference between the two probability density functions to the right of the debt level.
Figure 3: The figure shows the relation between default risk induced through the expected asset payment (Panel A), debt (Panel B) and volatility (Panel C & D) and the expected equity return. The effect of volatility on the expected equity return is examined for firms with a default risk below 50% (Panel C) and for firms with a default risk above 50% (Panel D). The base case parameters are: \( \rho \) is 0.95 and \( B \) is 0.05. \( C_0 \) and \( \hat{\mu}_C \) are 1 and 2, respectively. \( \hat{\sigma}_C \) is 1. \( \hat{\mu}_x \) and \( k \) are set to 1 (3 in panel D), while \( \sigma_x \) is 1.25. The correlation between the asset payment and consumption at time 1 is 0.60. In the figure, the relation between default risk and the expected equity return is examined under different time preferences and risk aversion parameters.
Figure 4: The figure shows the relation between default risk induced through the expected asset payment (Panel A), debt (Panel B) and volatility (Panel C & D) and the expected equity return. The effect of volatility on the expected equity return is examined for firms with a default risk below 50% (Panel C) and for firms with a default risk above 50% (Panel D). The base case parameters are: $\rho$ is 0.95 and $B$ is 0.05. $C_0$ and $\hat{\mu}_C$ are 1 and 2, respectively. $\hat{\sigma}_C$ is 1. $\mu_x$ and $k$ are set to 1 (3 in panel D), while $\sigma_x$ is 1.25. The correlation between the asset payment and consumption at time 1 is 0.60. In the figure, the relation between default risk and the expected equity return is examined under different expected consumption levels and correlations.