Implications of Predictability across Horizons for Asset Pricing Models

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Abstract

We study the properties of unconditional Hansen and Jagannathan (1991) bounds in the presence of conditioning information as the horizon increases. We provide evidence that long-horizon predictability translates into a tight lower bound on the variance of the stochastic discount factor (SDF). We then look at different asset pricing models and we show that all of them share a common feature at very long horizons: they have variance of the SDF that is too small compared to that in the data. The investment horizon is the fundamental dimension that allows the researches to set apart models, or, as in our case, to select the common behavior among apparently different models.
1 Introduction

In this paper we explore the role that short- and long-horizon predictability plays in the econometric evaluation of asset pricing models. In particular we link regression analysis at multiple horizons and bounds in the variance of stochastic discount factors (Hansen-Jagannathan bounds) to restrict the set of valid asset pricing models. Using the strategy first developed by Hansen and Jagannathan (1991) we show that the R of the predictive models for asset returns at different horizons imposes tight restrictions on the Sharpe ratio and therefore on the lower bound on the variance of the set of admissible stochastic discount factors (SDFs).

We first provide evidence that long-horizon predictability translates into a tight lower bound on the variance of the SDFs. This evidence hints at the importance of the choice of the conditioning information relevant for a specific investment horizon. We then examine a number of well-known asset pricing models: the long-run risk of Bansal and Yaron (2004) and its variant Bansal, Yaron, and Kiku (2010), the habit-formation of Campbell and Cochrane (1999) and its variant Bekaert and Engstrom (2010), the rare event of Backus, Chernov, and Martin (2011a) and an affine 3-factors model suggested by Koijen, Lustig, and Nieuwerburgh (2012) to explain the cross section of bond and stock returns. We show that all these models share a common feature at very long horizons: they have variance of the SDFs that is too small compared to that in the data. This is equivalent to state that these models have variance of the forecasts that is too small compared to that in the data. This conclusion is consistent with the idea that all models are approximations which necessarily are misspecified along some dimensions. The Hansen-Jagannathan bounds at multiple horizons identify the low frequency as the misspecified dimension. This is rather surprising given that most of the models that we consider incorporate a low frequencies component that should make asset pricing puzzle less pronounced at longer horizons.

Our multi-horizon Hansen-Jagannathan bounds represent a convenient tool for researchers: first they are informative of the dynamics that an admissible SDF should have at short-, medium- and long-term horizons; second, our bounds yield a graphical and intuitive comparison of the performance of asset pricing models by isolating the common feature underlying
them at a specific horizon. In fact, the dynamic asset pricing models under consideration are constructed from a mixture of assumptions about preferences (such as recursive utility or habit persistence, etc) and exposure to fundamental shocks. Our multi-horizon bound allows to abstract from these model ingredients and instead highlights the transitory and long-run implications of these economic models.

Our work is related to Kirby (1998), who provides an explicit link between linear predictability and Hansen and Jagannathan bounds. Our contribution refines his analysis and highlights the fundamental role played by the horizon dimension to deliver sharp restrictions on asset pricing models. Our work is also related to the recent literature which, using a decomposition of the model’s dynamics into transient and permanent components, investigates the implications of these components for valuation, see Hansen and Scheinkman (2009) and Borovicka, Hansen, Hendricks, and Scheinkman (2011). In particular we view our multi-horizon bounds as a methodology useful to understand the high- and low-frequency components of such models. Finally, our work is related to the recent information-theoretic literature that uses entropy bounds to restrict the admissible regions for the SDF and its components (see Bakshi and Chabi-Yo (2012) and Ghosh, Julliard, and Taylor (2011)). In particular our conclusions are in line with Backus, Chernov, and Zin (2011b) who shows that the entropy of a model should be sufficiently large to account for observed excess returns.

The rest of this paper is organized as follows. Sections 2.1 reviews the variance bounds of Hansen and Jagannathan (1991), concentrating on extensions for conditioning information (see Ferson and Siegel (2003) and Bekarert and Liu (2004)). Section 2.2 shows in a simple framework the dependence of Hansen-Jagannathan bounds on the holding period of returns. Section 3 documents the existence of significant predictable variation in stock and bond returns and shows how conditioning information plays an important role in the construction of the Hansen-Jagannathan bounds at different horizons. We then assess whether various specifications of SDFs are consistent with the evidence of multi-horizon Hansen-Jagannathan bounds uncovered using regression analysis. Section 4 concludes.
2 Variance Bounds, Predictability and Horizon

2.1 Variance Bounds and the Sharpe Ratio in the unconditional and conditional cases

In this section we introduce the basic notation and definitions, and we review both unconditional and conditional bounds on the variance of Stochastic Discount Factors (SDFs henceforth). Our review is based on extending the fundamental duality between minimum-variance SDFs and the maximum Sharpe ratio, see Hansen and Jagannathan (1991) (HJ henceforth) for the unconditional case, to the variance bounds based on conditioning information discussed in Gallant, Hansen, and Tauchen (1990) (GHT henceforth).

We consider an environment with $N$ returns traded at a given time $t$. We denote the return on each asset by $R_{j,t+h}$, where the $h = 1, 2,...$ stands for the investment horizon, and we let $R_{t+h}$ denote the vector collecting the $N$ returns. The underlying uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$, and the information sets available to investors at the times $t$, $t+h$, are formalized by the sigma algebras $\mathcal{F}_t \subseteq \mathcal{F}_{t+h} \subseteq \mathcal{F}$. We assume that returns have finite unconditional first and second moments, $\mu$ and $E^2 \triangleq E[R_{t+h}R'_{t+h}]$, with unconditional variance-covariance matrix $\Sigma = E^2 - \mu \mu'$. Moreover, we assume that the matrix $E^2_t \triangleq E[R_{t+h}R'_{t+h} / \mathcal{F}_t]$ of conditional second moments is (almost surely) positive-definite and hence invertible, so that returns are conditionally linearly independent. Denoting by $\mu_t$ the vector of conditional expected returns, this implies that both the conditional variance-covariance matrix $\Sigma_t = E^2_t - \mu_t \mu'_t$ and its unconditional counterpart $\Sigma$ are positive-definite (and hence invertible) as well.

We introduce two alternative sets of SDFs, $\mathcal{M}_1$, respectively $\mathcal{M}_2$, with the first set based on a weaker, unconditional restriction on returns and the second set based instead on a stronger, conditional moment restriction:

$$\mathcal{M}_1 = \{m_{t+h} \mid E(m^2_{t+h}) < \infty, \ E(m_{t+h}R_{t+h}) = e \}$$

and

$$\mathcal{M}_2 = \{m_{t+h} \in \mathcal{M}_1 \mid E[m_{t+h}R_{t+h} / \mathcal{F}_t] = e \}$$
where \( e \) denotes the unit vector. Modulo the restriction to returns, i.e. to unit-price assets, it is readily seen that the set \( \mathcal{M}_1 \) captures the SDFs that satisfy Restriction 1. in [Hansen and Jagannathan (1991)], while its subset \( \mathcal{M}_2 \) captures the SDFs that satisfy Restriction 2. in [Gallant et al. (1990)]. Intuitively, an SDF in \( \mathcal{M}_1 \) is required to satisfy the pricing equation only on average, while to belong to the subset \( \mathcal{M}_2 \) an SDF must satisfy the pricing restriction conditional on the events in the set \( \mathcal{F}_t \).

With this notation in place, we recall the definition for the variance bounds of the SDFs without and with conditioning information developed in [Hansen and Jagannathan (1991)] and [Gallant et al. (1990)], denoted by \( \sigma_{HJ}^2(\nu) \) and \( \sigma_{GHT}^2(\nu) \) respectively:

\[
\sigma_{HJ}^2(\nu) = \inf \{ \sigma^2(m_{t+h}) \mid m_{t+h} \in \mathcal{M}_1, E(m_{t+h}) = \nu \}
\]

and

\[
\sigma_{GHT}^2(\nu) = \inf \{ \sigma^2(m_{t+h}) \mid m_{t+h} \in \mathcal{M}_2, E(m_{t+h}) = \nu \}
\]

with \(-\infty < \sigma_{HJ}^2(\nu) \leq \sigma_{GHT}^2(\nu) < +\infty\) for any \( \nu \in \mathbb{R} \), where the parameter \( \nu \) is the (shadow) price of a riskless zero-coupon bond with maturity \( t+h \) and unit face value, so that when \( \nu \neq 0 \) then \( \nu^{-1} \) is a well-defined (shadow) risk-free rate. In particular, [Hansen and Jagannathan (1991)] show that the unconditional bound takes the parabolic shape

\[
\sigma_{HJ}^2(\nu) = a_{HJ} \nu^2 + b_{HJ} \nu + c_{HJ}
\]

where

\[
a_{HJ} = \mu'\Sigma^{-1}\mu, \quad b_{HJ} = -2\mu'\Sigma^{-1}e, \quad c_{HJ} = e'\Sigma^{-1}e
\]

The bounds obtained exploiting conditioning information are also parabolic,

\[
\sigma_{GHT}^2(\nu) = a_{GHT} \nu^2 + b_{GHT} \nu + c_{GHT}
\]

with coefficients that depend in this case from the conditional moments. In particular,
letting \( \alpha_t = e' \Sigma_t^{-1} e \), \( \beta_t = e' \Sigma_t^{-1} \mu_t \), \( \delta_t = (1 + \mu_t' \Sigma_t^{-1} \mu_t)^{-1} \) then

\[
a_{GHT} = [E(\delta_t)]^{-1} - 1, \quad b_{GHT} = -2E(\beta_t \delta_t), \quad c_{GHT} = [E(\delta_t)]^{-1} [E(\beta_t \delta_t)]^2 + E(\alpha_t) - E(\beta_t^2 \delta_t).\]

The inf problem that defines the variance bound has an important dual representation in terms of the Sharpe ratio. To see this, start with the unconditional case and define the set of returns (or, more generally, of payoffs with expected price equal to 1):

\[
\mathcal{R}_1 = \left\{ R_{t+h}^p \mid R_{t+h}^p = z'R_{t+h}, \ z \in \mathbb{R}^N \text{ s.t. } z'e = 1 \right\}.
\]

From the standpoint of portfolio formation, observe that in defining \( \mathcal{R}_1 \) the choice variable \( z \) is constant with respect to the information \( \mathcal{F}_t \) available at time. This way \( \mathcal{R}_1 \) represents the set of returns that are priced by the SDFs in \( \mathcal{M}_1 \). HJ show that, as long as \( \nu \neq 0 \) so that the shadow risk free rate \( \nu^{-1} \) is well-defined, the bound on the variance of the SDFs in \( \mathcal{M}_1 \) satisfies

\[
\sigma_{HJ}^2(\nu) = \nu^2 \sup_{R_{t+h}^p \in \mathcal{R}_1} \left( \frac{E(R_{t+h}^p) - \nu^{-1}}{\sigma(R_{t+h}^p)} \right)^2
\]

(1)

In words, the minimum variance across all SDFs that price the returns \( R_{t+h} \) unconditionally is proportional to the maximum Sharpe ratio from portfolio returns \( R_{t+h}^p \) formed with weights that are constant with respect to the information available at time \( t \).

Let’s turn now our attention to the case in which pricing is conditional on information, i.e. to the SDFs in the set \( \mathcal{M}_2 \) and to the bound on their variance. To extend the original

\[\text{1See e.g. Ferson and Siegel (2003).}\]

\[\text{2An alternative, equivalent expression for the coefficients } a_{GHT}, b_{GHT} \text{ and } c_{GHT} \text{ that involves the matrix of conditional second moments } E_t^2 \text{ instead of the conditional variance-covariance matrix } \Sigma_t \text{ is supplied by Bekaert and Liu (2004) who show that } \]

\[a_{GHT} = d/(1 - d), \quad b_{GHT} = -2b/(1 - d), \quad c_{GHT} = a + [b^2/(1 - d)]\]

where \( a = E\left[ e' \left( E_t^2 \right)^{-1} e \right], b = E\left[ e' \left( E_t^2 \right)^{-1} \mu \right] \) and \( d = E\left[ \mu_t' \left( E_t^2 \right)^{-1} \mu_t \right] \).

\[\text{3From standard mean-variance theory (see e.g. Huang and Litzenberger (1988), chapter 3) it follows that when } \nu = 0, \text{ denoting by } \sigma_{\min}^2 \text{ the minimum variance across all the returns in } \mathcal{R}_1, \text{ then } \sigma_{HJ}^2 = (\sigma_{\min}^2)^{-1}, \text{ i.e. the minimum variance across all SDFs that price the returns unconditionally and have zero mean is the reciprocal of the global minimum variance return in } \mathcal{R}_1. \text{ Also, the sup in (formula) is always attained in } \mathcal{R}_1 \text{ as long as } \nu \neq (E_{\min})^{-1}, \text{ where } E_{\min} \text{ is the expected return of the global minimum variance portfolio in } \mathcal{R}_1. \text{ If } \nu = (E_{\min})^{-1}, \text{ the sup is instead attained by a portfolio whose weights sum to zero, } z'e = 0, \text{ i.e. by a portfolio with zero expected price.}\]
duality result of HJ to this case, define the following set of returns:

\[ R_2 = \{ R_{t+h}^p \mid R_{t+h}^p = z_t' R_{t+h}, \ E[z_t' z_t] < \infty, \ \sigma^2(z_t' R_{t+h}) < \infty, \ E[z_t' e] = 1 \} . \]

The set \( R_2 \) collects all the payoffs that are generated by trading strategies that incorporate the information available at time \( t \). The payoffs in \( R_2 \), in particular, all have expected price equal to one, as can readily acknowledged by observing that

\[
E(m_{t+h} R_{t+h}^p) = E[z_t' E(m_{t+h} R_{t+h} / F_t)] = E[z_t' e] = 1, \ \forall m_{t+h} \in \mathcal{M}_2.
\]

As long as \( \nu \neq 0 \) one can show that

\[
\sigma^2(\text{GHT}(\nu)) = \nu^2 \sup_{R_{t+h}^p \in R_2} \left( \frac{E(R_{t+h}^p) - \nu^{-1}}{\sigma(R_{t+h}^p)} \right)^2
\]

This last result can be readily established by invoking Propositions 1. and 2. in Bekaert and Liu (2004), and observing that the function that they optimize to derive their optimally scaled bound is homogeneous of degree zero.

To conclude this review of the variance bounds that we will employ in our empirical analysis, we finally discuss the bound introduced by Ferson and Siegel (2003) (2003, FS henceforth). To this end consider the following subset \( R_3 \) of \( R_2 \):

\[
R_3 = \{ R_{t+h}^p \in R_2 \mid z_t' e = 1 \text{ almost surely} \}
\]

In words, the set \( R_3 \) collects all those payoffs that are generated by trading strategies that reflect the information available at time \( t \), and that have unit price almost surely equal to one, and not just on average. Given any \( \nu \neq 0 \), Ferson and Siegel (2003) propose the
following bound $\sigma^2_{FS}(\nu)$ on the variance of SDFs:

$$\sigma^2_{FS}(\nu) = \nu^2 \sup_{R^p_{t+h} \in \mathcal{R}_3} \left( \frac{E(R^p_{t+h}) - \nu^{-1}}{\sigma(R^p_{t+h})} \right)^2$$

To compute this bound, Ferson and Siegel (2003) compute any two distinct returns on the minimum variance in $\mathcal{R}_3$, that is $R^p_{t+h} \in \mathcal{R}_3$ such that $\sigma^2(R^p_{t+h}) \leq \sigma^2(R^p_{t+h})$ whenever $R^p_{t+h} \in \mathcal{R}_3$ and $E(R^p_{t+h}) = E(R^p_{t+h})$, $i = 1, 2$. Since the two-fund separation theorem extends to the case of stochastic portfolio weights (see Ferson and Siegel (2001)) and since to maximize the Sharpe ratio a portfolio needs to be on the frontier, the computation of $\sigma^2_{FS}(\nu)$ collapses to

$$\sigma^2_{FS}(\nu) = \nu^2 \sup_{\omega \in \mathbb{R}} \left( \frac{E(\omega R^p_{t+h} + (1 - \omega)R^p_{t+h}) - \nu^{-1}}{\sigma(\omega R^p_{t+h} + (1 - \omega)R^p_{t+h})} \right)^2$$

which in turn yields

$$\sigma^2_{FS}(\nu) = a_{FS}\nu^2 + b_{FS}\nu + c_{FS}$$

where

$$a_{FS} = \mu'_{FS}\Sigma^{-1}_{FS}\mu_{FS}, \quad b_{FS} = -2\mu'_{FS}\Sigma^{-1}_{FS}e, \quad c_{FS} = e'\Sigma^{-1}_{FS}e$$

with $\mu_{FS}$ the $2 \times 1$ vector collecting the means of $R^p_{t+h} \sqcup R^p_{t+h}$ and $\Sigma_{FS}$ the variance-covariance matrix of $R^p_{t+h} \sqcup R^p_{t+h}$.

It is now useful to compare this last variance bound with the ones introduced above. To do so, observe that since returns have finite unconditional first and second moments, then any deterministic trading strategy $z \in \mathbb{R}^N$ such that $z'e = 1$ produces a return in $\mathcal{R}_3$. As a consequence, $\mathcal{R}_1 \subset \mathcal{R}_3 \subset \mathcal{R}_2$, which in turn implies

$$\sigma^2_{HI}(\nu) \leq \sigma^2_{FS}(\nu) \leq \sigma^2_{GHT}(\nu)$$

In words, the variance bound introduced by FS is weaker than the GHT bound but stronger.
than the HJ bound. An SDFs in $\mathcal{M}_1$, that prices only the payoffs of unconditional strategies, is not required to have a volatility that satisfies the bound $\sigma_{FS}^2(\nu)$, and this is so because $\sigma_{FS}^2(\nu)$ is obtain as a max Sharpe ratio over returns that incorporate the information available at time $t$, and hence that are in general not priced by the SDFs in $\mathcal{M}_1$. Likewise, any SDF in $\mathcal{M}_2$, which needs to price returns conditionally, will clearly satisfy the Ferson and Siegel (2003) bound. As the duality approach discussed above clarifies, however, the Ferson and Siegel (2003) is a lower bound on the variance of the SDFs in $\mathcal{M}_2$, not the greatest lower bound. The reason for this is readily understood by observing that the returns in the set $\mathcal{R}_3$ are required to have conditional price equal to 1, while those in the set $\mathcal{R}_2$ are only required to have an expected price equal to 1.

We conclude this section by pointing out an important fact, first highlighted by Bekaert and Liu (2004), that clarifies the importance of the dual approach, in particular for the case of variance bounds that incorporate conditioning information. Recalling that $(\mu_t, \Sigma_t)$ denote the conditional first and second moments of the traded returns $R_{t+h}$, let $m_{t+h}^* [(\mu_t, \Sigma_t), \nu]$ represent the SFD in $\mathcal{M}_2$ with minimum variance given a mean $\nu$. Likewise, denote by $R_{t+h}^{p,BL} [(\mu_t, \Sigma_t), \nu]$ the solution to the corresponding dual problem, i.e. $R_{t+h}^{p,BL} [(\mu_t, \Sigma_t), \nu]$ represents the return that maximizes the Sharpe ratio over the set of managed returns $\mathcal{R}_2$, given a (shadow) risk-free rate $\nu^{-1}$. Suppose now that the econometrician mis-specifies the conditional moments of the returns distribution, i.e. suppose that he operates under the assumption that the conditional moments are $(\tilde{\mu}_t, \tilde{\Sigma}_t) \neq (\mu_t, \Sigma_t)$. As long as $m_{t+h}^* [(\tilde{\mu}_t, \tilde{\Sigma}_t), \nu] \in \mathcal{M}_2$ and $R_{t+h}^{p,BL} [(\tilde{\mu}_t, \tilde{\Sigma}_t), \nu] \in \mathcal{R}_2$, since $\sigma^2 \{ m_{t+h}^* [(\mu_t, \Sigma_t), \nu] \} \equiv \sigma_{GHT}^2(\nu) \equiv \sigma^2 \{ R_{t+h}^{p,BL} [(\mu_t, \Sigma_t), \nu] \}$, the very definition of inf and sup implies that

$$\sigma^2 \{ m_{t+h}^* [(\tilde{\mu}_t, \tilde{\Sigma}_t), \nu] \} \geq \sigma_{GHT}^2(\nu) \geq \sigma^2 \{ R_{t+h}^{p,BL} [(\tilde{\mu}_t, \tilde{\Sigma}_t), \nu] \}$$

These inequalities establish the following fundamental fact, pointed out by Bekaert and Liu (2004): if the model for the conditional moments is mis-specified, then the primal problem, analyzed by Gallant et al. (1990), may lead to reject SDFs that are inside the 'true' bound. The dual problem, on the contrary, represents a lower bound on the variance of an SDF, even when the conditional moments are mis-specified although in this case it
would not yield the highest lower bound.

2.2 Predictability and Variance Bounds

The available empirical literature on the predictability of returns (see Cochrane (2008), Campbell and Viceira (2005)) shows that returns predictability increases with the holding period (i.e. the time horizon during which the portfolio is unchanged) and that the term structure of volatility is downward sloping. This evidence is consistent with a decomposition of returns into an highly volatile and unpredictable "noise" component, and a low-volatility, predictable "information" component. The dominance of the "noise" component in short-horizon returns and of the "information" component in long horizon returns implies a positive relation between predictability of returns and forecasting horizon and a negatively sloped term structure of risk.

Consider the following representation of predictive regressions for returns at different horizons:

\[ R_{t+h} = \beta_0 + \beta_1 P_t + v_t \]

\[ v_t \sim (0, \Sigma) \]

\[ \mu_t = \beta_0 + \beta_1 P_t \]

where \( P_t \) is a vector of predictors for returns observable at time \( t \). The conditional first and second moments of traded returns are \((\mu_t, \Sigma)\). The empirical evidence available tells us that the performance of model (4) in predicting returns improves with the horizon. In the short horizon there is virtually very little difference between conditional and unconditional moments as the predictability is negligible, however as the forecasting horizon increases some sizable predictable materializes. As illustrated in the previous section such predictability drives a difference between unconditional and conditional HJ bounds.

To illustrate the role of noise and fundamentals in determining a term-structure of predictability and its importance to drive a wedge between unconditional and conditional HJ bounds consider the following illustrative model where returns are determined by the dy-
namic dividend growth model (see Campbell and Shiller (1988a), Campbell and Shiller (1988b)):

\[ r^S_{t+1} = \rho_0 - \rho (d_{t+1} - p_{t+1}) + \Delta d_{t+1} + (d_t - p_t) \quad (5) \]

\[ \Delta d_{t+1} = a \Delta d + \sigma_1 \varepsilon_{1,t+1} \quad (6) \]

\[ (d_{t+1} - p_{t+1}) = a dp + \phi (d_t - p_t) + \sigma_2 \varepsilon_{2,t+1} \quad |\phi| < 1 \quad (7) \]

Equation (5) specifies the linearized definition of returns to which, for simplicity, we add no stochastic error. Equation (6) specifies the process for the dividend growth as a white noise, where we label \( \varepsilon_{1,t+1} \) the innovation to real dividend growth. This simple parameterization is fully consistent with the evidence of very little predictability of dividend growth. Equation (7) specifies the process for the dividend-price ratio as an autoregressive process whose innovations capture noise, the dividend price ratio converges slowly to a long-run mean determined by fundamentals but it is dominated by noise over the short run. The conditional volatility of this process is time-varying but the unconditional volatility is constant. The one-step ahead prediction of this model for stock market returns can be written as:

\[ r^S_{t+1} = -\rho a dp + a + (1 - \rho \phi) (d_t - p_t) + v_{t+1} \quad (8) \]

\[ v_{t+1} = \sigma_1 \varepsilon_{1,t+1} - \rho \sigma_2 \varepsilon_{2,t+1} \]

One-period ahead returns are dominated by the noise components that determines a time-varying volatility and very little predictability of future returns given current fundamentals. The \( R^2 \) of the projection of one-period returns at time \( t+1 \) on the predictors at time \( t \) is very low and there is virtually no-difference between the conditional and the unconditional expectation \( r^S_{t+1} \) and its conditional and unconditional variance. The HJ bounds computed conditionally and unconditionally will therefore virtually coincide.

Iterating forward for \( h \) periods, we obtain simple model for the determination of long-run

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6 We assume, for simplicity, an homoscedastic process for noise. The model can be easily extended to allow for time-varying volatility. In this case it can be shown that also heteroscedasticity is a function of the horizon and it gradually disappears upon aggregation of high-frequency returns in low frequency returns.
stock market returns:

\[
\sum_{j=0}^{h} \rho^j r^S_{t+j+1} = (d_t - p_t) + \sum_{j=0}^{h} \rho^j (\Delta d_{t+j+1} - \rho^{h+1} (d_{t+h+1} - p_{t+h+1})
\]

\[
(d_{t+h+1} - p_{t+h+1}) = \phi^{h+1} (d_t - p_t) + \sum_{j=0}^{h} \phi^j \sigma d_{t+h+1+j} + \sum_{j=0}^{h} \phi^j \sigma \epsilon_{2,t+1+j}
\]

The model states clearly that, in absence of bubbles \( \lim_{h \to \infty} \rho^h (p_t - d_t) = 0 \), expected long-run stock market returns \( E_t \sum_{j=0}^{h} \rho^j \left( r^S_{t+j+1} \right) \) depend on the current dividend-yield and on future expected dividend growth \( E_t \sum_{j=0}^{h} \rho^j (\Delta d_{t+j+1}) \). As a consequence the importance of noise to determine short-run returns disappears gradually with the horizon at which returns are defined and fundamentals take progressively over.

As the horizon \( h \) at which returns are defined gets larger, the importance of fundamentals is reflected in an increased significant of the predictor and in an increased \( R^2 \) of the predictive regression. As a consequence the conditional expectation will differ from the unconditional one and the unconditional variance will be higher than the conditional one. The wedge between the unconditional and the conditional variance will be at the maximum when the information set used by the econometrician and the agents coincide (no omitted variable in the econometric specification). As a consequence the longer the horizon the larger the difference between the variance bounds computed using conditional and unconditional moments.

To see this point consider the predictive regressions for the \( h \)-period ahead returns

\[
\sum_{j=0}^{h} \rho^j (R^S_{t+j+1}) = \left( 1 - (\rho \phi)^{h+1} \right) (d_t - p_t) + \sum_{j=0}^{h} \rho^j a_{\Delta d} - \rho^{h+1} \sum_{j=0}^{h} \phi^j a_{dp}
\]

\[
+ \sum_{j=0}^{h} \rho^j \sigma_{1,t+j+1} - \rho^{h+1} \sum_{j=0}^{h} \phi^j \sigma_{2,t+1+j}
\]

In our simple example, the term structure of stock market risk takes the form
Given that $|\phi| < 1$ the forecasting variance of the predictive regression scaled by the investment horizon will be decreasing with the horizon, generating an increasing difference between the variance bounds computed on the conditional and unconditional moments.

Interestingly, the estimation of a bivariate VAR model for the predictor and the return will deliver a different term structure of risk than the direct regression, as the bivariate VAR will be estimating a backward-looking approximation to the true forward-looking model. To see this point, we specify a VAR model consistent with our DGP, see (7) and (8), and we show that the variance of the period returns based on the VAR will be larger than that based on the direct estimator. Formally

$$(z_t - E_z) = \Phi_1 (z_{t-1} - E_z) + u_t$$

where

$$z_t = \begin{bmatrix} r_t^s \\ dpt \end{bmatrix}, E_z = \begin{bmatrix} E_{r^s} \\ E_{d-p} \end{bmatrix}, \Phi_1 = \begin{bmatrix} 0 & 1 - \rho \phi \\ 0 & \phi \end{bmatrix}$$

and the innovations are

$$u_t = \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 + \rho^2 \sigma_2^2 & -\rho \sigma_2^2 \\ -\rho \sigma_2^2 & \sigma_2^2 \end{bmatrix} \right)$$

Given the VAR representation and the assumption of constant $\Sigma$, it is possible to show that in our simple bivariate example, the term structure of stock market risk takes the form\(^7\)

\(^7\)See Appendix 5 for the derivation.
\begin{equation}
\text{Var} \left[ \sum_{j=0}^{m} \rho^j \left( R_{t+j+1}^e \right) \mid \mathcal{F}_t \right] = \sum_{j=0}^{m} \rho^{2j} \left[ \sigma_1^2 + \sigma_2^2 \rho^2 + 2 \left( -\rho \sigma_2^2 \right) (1 - \rho \phi) \sum_{i=0}^{j-1} \phi^i + \sigma_2^2 (1 - \rho \phi)^2 \left( \sum_{i=0}^{j-1} \phi^i \right)^2 \right]
\end{equation}

When \( \phi = 0 \), i.e. returns is unpredictable, (9) and (10) yield the same result at each horizon. Once there is predictability of returns, the term structure of stock market risk would be different, particularly at long horizons. The differences between (9) and (10) only depend on the coefficients of \( \sigma_2^2 \), \( \rho^2 (m+1) \sum_{j=0}^{m} \phi^{2j} \) and \( \sum_{j=0}^{m} \rho^{2j} \left( \rho - (1 - \rho \phi) \sum_{i=0}^{j-1} \phi^i \right)^2 \). Now we study the behavior of the two coefficients across horizon by giving a simple numerical example. Given \( \rho = 0.9 \) and \( \phi = 0.9 \), Figure 1 plots the value of \( \sum_{j=0}^{h} \rho^{2j} \left( \rho - (1 - \rho \phi) \sum_{i=0}^{j-1} \phi^i \right)^2 \) across horizon.\(^8\) The variance of the holding period returns based on the VAR model will be larger than that based on the direct estimation. In particular, with horizon increasing, the differences increases. Hence, the conditional variance bounds of SDFs would be tighter once we use direct regression instead of a VAR model.

[Insert Figure 1 about here]

### 3 Empirical Results

This section highlights the role of conditional information to construct the HJ bounds. In the context of a finite number of assets we show that the efficient use of conditioning information is particularly relevant to sharpen the bounds at long horizons. Then we elaborate on the performance of asset pricing models in satisfying the HJ bounds.

\(^8\)Based on US market data with annual frequency, the autocorrelation coefficient of dividend-price ratio is about 0.9 and the value of parameter \( \rho \) is also around 0.9. See Campbell and Shiller (1988a) and Cochrane (2005).
3.1 Description of the set of asset returns

We consider a set of quarterly equity and bond returns over the period 1952Q2 to 2010Q4. Our choice of the start date is dictated by the availability of data for our predictors. The empirical analysis is based on the (gross) return on the constant maturity bond, and on the (gross) return on the value weighted portfolio of all stocks traded in the NYSE, the AMEX, and NASDAQ. Real returns are computed by deflating nominal returns by the Consumer Price Index inflation. We obtain the time series of bond and stock returns using monthly daily returns on stocks and bonds extracted from the US Treasuries and Inflation Indices File and the Stock Indices File of the Center of Research in Security Prices (CRSP) at the University of Chicago. The CRSP US Treasuries and Inflation Indices File provides returns on constant maturity coupon bonds, with maturities ranging from 1 year to 30 years, starting on January, 1942. Quarterly returns are constructed by compounding their monthly counterparts. Table 1 presents full-sample statistics of the the 5-year constant maturity bond and stock returns for the common sample period (1952Q2 to 2010Q4). Over this sample period, the mean log return on stocks was 9.88% per annum, the mean log return on bonds was 6.38% per annum, and the mean short-term interest rate - not shown in the table - was about 5.65% per annum. The standard deviation of stock log returns was 16.86% p.a., and the standard deviation of bond log returns was 5.81% p.a.

We consider two different set of assets:

1. SET A: risk-free bond, 5-year Treasury constant maturity and equity market returns
2. SET B: risk-free bond, 5-, 7-, 10- and 30-year Treasury constant maturity government bonds and equity market returns.

These two sets correspond to a universe of equity and bond portfolios whose return properties are the subject of much scrutiny in the empirical asset pricing research.

9See also the Appendix 5 for a detailed description of data construction.
3.2 Multi-Horizon Bounds without Information

We first explore the sensitivity of the HJ bounds to the “investment horizon” between now and then periods when no conditional information is used. We consider investment horizons of 1 quarter, 1 year, and 5 years. The shortest investment horizon coincides with the sampling interval of returns. Long-horizon returns are computed by compounding quarterly returns. This exercise is in the same spirit of Cochrane and Hansen (1993) who study the effect of altering the investment horizon on the unconditional variance bounds. Figure 2 reports the HJ regions for 1-quarter, 1-, and 5-year horizons.

Figure 2 shows that the bottom of the mean standard deviation frontier shifts up and to the left as we increase the investment horizon. At the same time, although the lower bound for volatility increases, it does so slowly. There is also less information about the mean of longer horizon discount factors, as reflected by the horizontal expansion of the regions. We next explore how this bounds change if we account for conditional information.

3.3 Multi-Horizon Bounds with Conditioning Information

The predictability of stock and bond returns has received a lot of attention in the finance literature (see Fama and French (1989), Campbell (1987) and Cochrane (2001) among others). The objective of this section is to show how the predictable variation in returns can be used to sharpen the unconditional volatility bounds.

Bekaert and Liu (2004) shows that the optimal trading strategy $z_t$ that incorporates information takes the following expression:

$$z_t = (\mu_t \mu_t^\top + \Sigma_t)^{-1} (1 - w) \mu_t$$

where $w = (\nu - b)/(1 - d)$, $\nu = E[M_{t+1}]$, $b = E \left[ c' (\mu_t \mu_t^\top + \Sigma_t)^{-1} \mu_t \right]$ and $d = E \left[ \mu_t' (\mu_t \mu_t^\top + \Sigma_t)^{-1} \mu_t \right]$. Hence to compute $\sigma^2 \left\{ R_{t+h}^{p, BL} \left[ (\tilde{\mu}_t, \tilde{\Sigma}_t), \nu \right] \right\}$, see [2], we need a model of the first and second conditional moments of asset returns, $\mu_t$ and $\Sigma_t$. Our model for stock returns is formed from regressions of the returns on the dividend-price ratio and $cay$, which is a log lineariza-
The model for bonds is formed from regressions of returns onto the lagged short-term nominal interest rate, the lagged yield spread and the Cochrane-Piazzesi (CP) factor, see Cochrane and Piazzesi (2005). The yield spread is the log yield on a 5-year artificial zero-coupon bond from the CRSP Fama-Bliss Discount Bond File minus the short-term rate.

\begin{align}
R_{t,t+h}^B &= \rho_{0,h}^B + \rho_{1,h}^B s_{pr,t} + \rho_{2,h}^B y_t + \rho_{3,h}^B CP_t + u_{t,t+h}^B \\
R_{t,t+h}^S &= \rho_{0,h}^S + \rho_{1,h}^S pd_t + \rho_{2,h}^S cay_t + u_{t,t+h}^S
\end{align}

where $h = 1, 4, 20$ quarters.

Table 2 displays the results for the above linear predictive regressions. The table reports the slopes, the $R^2$ of the regressions, and the Newey-West corrected t-statistic of the significance of the slopes. Table 2 - Panel A presents regressions of stock log excess returns. The results are consistent with much of the recent empirical research and show that the price-dividend ratio together with the consumption-wealth ratio track variations in stock returns measured over several years. Table 2 - Panel B presents regressions of bond log excess returns, measured at horizons ranging from one quarter to five years, onto the lagged short-term nominal interest rate and the lagged yield spread. Bond excess returns are measured as the excess log total return on a constant 5-year maturity Treasury coupon bond over the short-term nominal interest rate, taken as the log yield on a 30-day Treasury Bill. Table 2 shows that our predictors are able to capture fluctuations in bond excess returns at all horizons.

[Insert Table 2]

We now revisit the HJ bounds when conditional information is used and we show how the strong predictability at long horizons documented in Table 2 translates into a tight lower bound. Our results are not driven by the inclusion of the CP factor. In a system where the CP factor is not included the Newey-West corrected t-statistic on the yield spread is above 2.2 at all horizons and the predictive system achieves a high $R^2$ value of 44.8% at the five year horizon.
bound on the variance of the SDF. Figure 2 imparts three conclusions. First, the estimates of HJ bounds generated from SET A and using efficiently conditional information (solid lines) are sharper relative to the unconditional ones (dashed lines). In particular Figure 2 shows that the minimum point of the frontier for the volatility of SDF at the 5-year horizon based on the return properties of SET A (SET B) and using conditional information is about 1.6 (1.4) times sharper than the unconditional lower bound, thereby substantiating the incremental value of conditional information in asset pricing applications. The difference between the bounds with and without conditional information at the 5-year horizon reflects the considerable long-run predictability documented in Table 2. Second, the implications of using conditional information strongly depend on the holding period of returns: the use of conditional moments adds little at short-horizon; at the same time the volatility implications of the long-horizon returns are more dramatic and reveal the fundamental role played by conditional information. Consistently with the results on the predictive regressions, the role of information becomes apparent as we lengthen the investment horizon. The figure highlights the two effects that are at work simultaneously: the conditioning information embedded in the conditional first and second moments of returns and the horizon at which this information becomes relevant. The tightening of the volatility bounds is a result of these two forces. Finally, comparing Panel A to Panel B, the HJ bounds declines from SET B to SET A, since expanding the number of assets leads to a bound that is intrinsically tighter.

Our multi-horizon bounds seems to be a useful tool to assess the performance of candidate asset pricing models at multiple horizons. Importantly recent theoretical and empirical research in macro-finance has highlighted the importance of modeling low frequencies components for an asset pricing model to be successful. One would then expect this low frequency component to make asset pricing puzzle less pronounced at longer horizons. We show next that the opposite happens.

3.4 Simulation Results

The objective of this section is to compare the stochastic discount factor dynamics implied by different asset pricing models. In particular we examine whether the variance of the
$h$-period SDF implied by a specific model respects the HJ variance bounds at the corresponding horizon. This analysis gives insights into how economic fundamentals are linked to the SDFs, and how the performance of an asset pricing model could be improved by altering the properties of SDFs. Specifically, we focus on models in the long-run risk class (Bansal and Yaron (2004) and BKY09), the external habit persistence class (Campbell and Cochrane (1999) and Bekaert and Engstrom (2010)), and the rare consumption disasters class (Rietz (1988) and Barro (2006)). Primitive parameters are chosen consistently according to the models of Bansal and Yaron (2004) and Bansal et al. (2010), Campbell and Cochrane (1999) and Bekaert and Liu (2004), and Backus et al. (2011a), respectively. The parameters are displayed in the following tables. We also consider a recent affine model suggested by Koijen et al. (2012) where 3-factors, the level of interest rates, $CP$ and the price-dividend explain the cross section of bond and stock returns. Although the list of models under consideration is far from exhaustive, they still embed different utility specifications and specify the long-run and short-run risk in distinct ways (see also Hansen (2009)).

[Insert Tables 9, 10 and 11 about here: parameters of models]

Each of the above mentioned models is calibrated closely to the mean and standard deviation of consumption growth, and offer conformity with the historical real return of risk-free bond and the real return of the equity market. Our hope is that the multi-horizon bounds could serve as a differentiating diagnostic for competing asset pricing models. While the conditional variances are amenable to closed-form characterization, the unconditional variances are tractable only via simulations\footnote{The only exception is for the rare disaster model given i.i.d. uncertainties.}. Therefore to assess the ability of a model to produce realistic SDF dynamics we rely on a statistic generated from a simulation procedure and the associated $p$-value. In particular each model is simulated using the dynamics of consumption growth and other state variables over a single simulation run of 360,000 months (30,000 years). Then we build the time series of model-specific SDF, and we calculate the unconditional moments. Using a single simulation run to infer the population values for the entities of interest is consistent with, among others, the approach of Campbell and Cochrane (1999) and Beeler and Campbell (2009).
Table 3 reports, for each model, the variance of the SDF. Reported in the final column are the lower bounds calculated based on SET A and SET B respectively.

Our implementations reveal that the monthly standard deviation implied by the Bansal and Yaron (2004), Bekaert and Engstrom (2010) and Backus et al. (2011a) models are 1.43, 1.41 and 1.03, respectively. All these values are lower than 1.44, the minimum volatility suggested by the HJ bound constructed from SET A. The best performing model are the Bansal, Kiku, and Yaron (2007a) with a volatility of 1.95 and the Campbell and Cochrane (1999) model with a 1.54, respectively. Although these values are pronounced they are still lower than the minimum volatility obtained from the SET B.

Going further, we formulate the restriction: \( \sigma^2 \left\{ R_{p,BL}^{t+h} \left[ (\tilde{\mu}_t, \tilde{\Sigma}_t), \nu \right] \right\} - \text{Var}(M_{t,t+h}) < 0 \) for a candidate asset pricing model, which allows one to elaborate on whether a model respects the lower bound (beyond eye-balling estimates; see also Cecchetti, Lam, and Mark (1994)). Then inference regarding this restriction can be drawn via repeated simulations (e.g., Patton and Timmermann (2010)). For this purpose, we rely on a finite-sample simulation of 700 months (1952:06 to 2010:9), and we choose 200,000 replications. The proportion of the replications satisfying \( \sigma^2 \left\{ R_{p,BL}^{t+h} \left[ (\tilde{\mu}_t, \tilde{\Sigma}_t), \nu \right] \right\} - \text{Var}(M_{t,t+h}) < 0 \) can be interpreted as a p-value for a one-sided test. This p-value is shown in curly brackets in Tables 4 and 5, for SET A and SET B respectively, and a low p-value indicates rejection. The final column is the lower bound calculated based on SET A (SET B), along with the 90% confidence intervals, shown in square brackets, from a block bootstrap, when sampling is done with ?? blocks. Importantly in order to account for the possible model and parameters uncertainty we compare the SDF volatility from a simulated model with 90% confidence interval of the HJ frontier. Pertinent to this exercise, our evidence reveals that the variance of the SDF from almost every model fails to meet the restrictions imposed by the HJ bounds at horizon \( H = 5 \)–years. The highest p-value of 0.052 corresponds to the model of Bansal et al. (2010). This is consistent with the long-run simulation exercise reported in Table 3. Judging by these results, the approach in Bekaert and Engstrom (2010) does not appear...
to substantially improve upon the variance of the permanent component relative to \cite{Campbell1999}. This finding is somewhat unexpected, given a flexible modeling of consumption growth in conjunction with a modified process for marginal utility. The fact that all these models do not differ markedly in their capacity to generate a volatile long-horizon SDF comes at a surprise. Despite the different assumptions about preferences (such as risk aversion or habit persistence, etc) and exposure to unforeseen shocks underlying these dynamic models, they all have the same long-run implications. The conclusion that emerges when the volatility of SDF is computed from SET B, see Table 5, exacerbates the results. In this case our evidence reveals that the variance of the SDF from each model fails to meet the lower bound restriction at 5-year horizon.

[Insert Tables 4 and 5 about here]

It is noteworthy that each asset pricing model parametrization reasonably mimics the (annual) equity premium and the real risk-free return, while simultaneously calibrating closely to the first-two moments of consumption growth. Thus, there appears to be a tension, within a model, between matching the annual sample average of equity returns and risk-free returns, versus generating a minimum volatility of the SDF at longer horizons. To sum up as the investment horizon increases, the Equity Premium puzzle does not vanish, but instead appear to be more pronounced. In the extreme case of a 5-year investment horizon and SET B, the standard deviation of the SDF never approaches the bounds. Models may reproduce some asset market phenomena at short horizon, but find it onerous to satisfy bounds at longer ones.

4 Robustness

This section starts by illustrating the performance of our bound along two dimensions: robustness and efficiency. In fact note that our results are obtained by assuming a time-invariant variance-covariance matrix for returns and a linear model for their conditional mean. To investigate possible mis-specification of the conditional moments and the efficiency of our bound we plot in Figure 3, alternative implementations of the HJ bounds: in particular

[Insert Figure 3 about here]

Bekaert and Liu (2004) shows that the BL bound should be a parabola if the moments are correctly specified. Figure 3 shows that indeed we obtain a smooth parabola. Moreover the figure reveal that the GHT and optimally scaled bound are virtually on top of one another. This suggests that the BL bound closely approximate the efficient use of the conditioning information. Overall the three alternative implementations of the HJ bounds generate similar bounds with no visible misspecification. The FS is the lowest bound: this is consistent with [3], i.e. with the fact that this bound collects all those payoffs that are generated by trading strategies that reflect the information available at time $t$, and that have unit price almost surely equal to one, and not just on average as for the BL case.

We construct confidence intervals for the Hansen-Jagannathan bounds. Results show that our values are somehow conservative and that using the 10% confidence interval bound can exacerbate our conclusions.

[Place Figures 4(a) and 4(b) about here]

We have seen that the FS bounds provide a conservative one. Tables 6 and 7 show that our conclusions do not change when we construct the bounds using their approach.

[Insert Tables 6 and 7 about here]

Our exercise shows that the long-run risk of Bansal et al. (2010) turns out to be the most successful model. We check if the results are robust to different choice of the coefficient of relative risk aversion and elasticity of intertemporal substitution.

[Insert Table 8 about here]

The results show that the bounds can only be informative about the degree of risk-aversion; an high degree of risk aversion is needed for the model to enter the long-horizon bounds.
5 Conclusion

We propose multi-horizon HJ bounds as a useful tool to assess the performance of an asset pricing model at different horizons. We have shown that long-horizon predictability translates into a large lower bound on the variance of the SDFs: in particular the $R^2$ of the predictive models for asset returns which increases with the predictive horizon imposes tight restrictions on the Sharpe ratio and therefore on the lower bound on the variance of the set of admissible stochastic discount factors (SDFs).

We have then examined a number of well-known asset pricing models: the long-run risk of Bansal and Yaron (2004) and its variant Bansal et al. (2010), the habit-formation of Campbell and Cochrane (1999) and its variant Bekaert and Engstrom (2010), the rare event of Backus et al. (2011a) and an affine 3-factors model suggested by Koijen et al. (2012) to explain the cross section of bond and stock returns. We show that all these models share a common feature at very long horizons: they have variance of the SDFs that is too small compared to that in the data. It is noteworthy that each asset pricing model parametrization reasonably mimics the (annual) equity premium and the real risk-free return, while simultaneously calibrating closely to the first-two moments of consumption growth. Thus, there appears to be a tension, within a model, between matching the annual sample average of equity returns and risk-free returns, versus generating a minimum volatility of the SDF at longer horizons. To sum up as the investment horizon increases, the Equity Premium puzzle does not vanish, but instead appear to be more pronounced. In the extreme case of a 5-year investment horizon and an investment set comprising a risk-free bond, 5-, 7-, 10- and 30-year Treasury constant maturity government bonds and equity market returns, the standard deviation of the SDF never approaches the bounds. Models may reproduce some asset market phenomena at short horizon, but find it onerous to satisfy bounds at longer ones.
Data

Quarterly Dataset:

1. Returns: market index, T-bill and T-bond. We use the NYSE/Amex value-weighted index with dividends as our market proxy, $R_{t+1}$. Return data on the value-market index are obtained from the Chicago Center for Research in Security Prices (CRSP). The nominal short-term rate ($R_{f,t+1}$) is the annualized yield on the 3-month Treasury bill taken from the CRSP treasury files. The $h$-horizon continuously compounded excess return is calculated as $r_{t,t+h} = r_{t+1}^e + \ldots + r_{t+h}^e$ where $r_{t+j}^e = \ln(R_{t+j}) - \ln(R_{f,t+j})$ is the 1-year excess log stock return between dates $t+j-1$ and $t+j$; $R_{t+j}$ is the simple gross return; and $R_{f,t+j}$ is the gross risk-free rate (3-month Treasury bill) at the beginning of period $t+j$.

Returns on bonds are extracted from the Fixed Term Indices File of the Chicago Center of Research in Security Prices (CRSP). The CRSP Fixed Term Indices File provides daily total returns and yields on constant maturity coupon bonds, with maturities ranging from 1 year to 30 years, starting on June 14, 1961.


3. Inflation: we use the seasonally unadjusted CPI from the Bureau of Labor Statistics. Quarterly inflation is the log growth rate in the CPI.

The term-structure of risk implied by a VAR model

Given the VAR representation and the assumption of constant $\Sigma$

$$
Var \left[(z_{t+1} + \rho z_{t+2} \ldots + \rho^m z_{t+m+1}) \mid \mathcal{F}_t \right] = \Sigma + \rho^2 (I + \Phi_1) \Sigma (I + \Phi_1)' \\
+ \rho^4 (I + \Phi_1 + \Phi_1^2) \Sigma (I + \Phi_1 + \Phi_1^2)' + \ldots \\
+ \rho^{2m} (I + \Phi_1 + \ldots + \Phi_1^m) \Sigma (I + \Phi_1 + \ldots + \Phi_1^m)' 
$$
from which we can rewrite:

\[
Var \left[ (z_{t+1} + \rho z_{t+2} + \ldots + \rho^m z_{t+m+1}) \mid F_t \right] = \sum_{j=0}^{m} \rho^{2j} D_j \Sigma D'_j
\]

\[
D_j = I + \Phi_1 \Xi_{j-1} \quad j > 0
\]

\[
\Xi_j = \Xi_{j-1} + \Phi_1^j \quad j > 0
\]

\[
D_0 \equiv I, \quad \Xi_0 \equiv I
\]

Note that, under the chosen specification of the matrix \(\Phi_1\) we can write the generic term \(D_j \Sigma D'_j\), as follows:

\[
D_j \Sigma D'_j = \begin{pmatrix} M_{11} & M_{12} \\ M'_{12} & M_{22} \end{pmatrix}
\]

\[
M_{11} = \Sigma_{1,1} + \Phi_1^{(12)} \Xi_{j-1} \Sigma_{{1,2}} + \Sigma_{{1,2}} \Xi_{j-1} \Phi_1^{(12)} + \Phi_1^{(12)} \Xi_{j-1} \Sigma_{2,2} \Xi_{j-1} \Phi_1^{(12)}
\]

\[
M'_{12} = \Xi_j^{(22)} \Sigma_{{1,2}} + \Xi_j^{(22)} \Sigma_{{2,2}} \Xi_{j-1} \Phi_1^{(12)}
\]

\[
M_{22} = \Xi_j^{(22)} \Sigma_{{2,2}} \Xi_j^{(22)}
\]

where we have used the fact that

\[
\Xi_j = \sum_{i=0}^{j} \Phi_1^i
\]

\[
= \begin{pmatrix} 0 & \Phi_1^{(12)} \sum_{i=0}^{j-1} \left( \Phi_1^{(22)} \right)^i \\ 0 & \sum_{i=0}^{j} \left( \Phi_1^{(22)} \right)^i \end{pmatrix}
\]
and

\[
D_j = I + \Phi_1 \Xi_{j-1}
\]

\[
= \begin{pmatrix}
I & \Phi_1^{(12)} \sum_{i=0}^{j-1} \left( \Phi_1^{(22)} \right)^i \\
0 & \sum_{i=0}^{j} \left( \Phi_1^{(22)} \right)^i
\end{pmatrix}
\]

Equation (13) implies that, in our simple bivariate example, the term structure of stock market risk takes the form

\[
\text{Var} \left[ \sum_{j=0}^{m} \rho^j (R_{t+j+1}^e) \left| \mathcal{F}_t \right. \right]
\]

\[
= \sum_{j=0}^{m} \rho^{2j} \left[ \sigma_1^2 + \sigma_2^2 \rho^2 + 2 (-\rho \sigma_2^2) (1 - \rho \phi) \sum_{i=0}^{j-1} \phi^i + \sigma_2^2 (1 - \rho \phi)^2 \left( \sum_{i=0}^{j-1} \phi^i \right)^2 \right]
\]

\[
= \sigma_1^2 \sum_{j=0}^{m} \rho^{2j} + \sigma_2^2 \sum_{j=0}^{m} \rho^{2j} \left( \rho - (1 - \rho \phi) \sum_{i=0}^{j-1} \phi^i \right)^2, \quad m > 0
\]
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Figure 1 Figure 1 plots the value of \( \sum_{j=0}^{h} \rho^{2j} \left( \rho - (1 - \rho \phi) \sum_{i=0}^{j-1} \phi^i \right)^2 - \rho^{2(h+1)} \sum_{j=0}^{h} \phi^{2j} \) across horizon.
Figure 2 Volatility bounds on stochastic discount factors for different investment horizons. Dashed line gives the volatility bound when no conditional information is used. Solid line gives the volatility bound using conditional information. The bounds are generated using SET A (see Panel A) and SET B (see Panel B). Long horizon returns are computed by compounding quarterly returns.
Figure 4 Volatility bounds on stochastic discount factors and confidence intervals for different investment horizons. Dashed line gives the volatility bound when no conditional information is used. Solid line gives the volatility bound using conditional information. The bounds are generated using SET A (see Panel A) and SET B (see Panel B). Long horizon returns are computed by compounding quarterly returns.
Table 1 This table reports sample statistics of quarterly log stock and bond total returns. Stock returns are log returns on the stock total returns on the value weighted portfolio of all stocks traded in the NYSE, the AMEX, and NASDAQ from CRSP. Bond returns are log returns on the 5-year constant maturity bond from the CRSP Fixed Term Indices File.

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<th>Asset</th>
<th>Stocks</th>
<th>Bonds</th>
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<tr>
<td>Mean return (% p.a.)</td>
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<td>6.38</td>
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<tr>
<td>Standard deviation (% p.a.)</td>
<td>16.86</td>
<td>5.81</td>
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Panel A: Predictive regressions for stock returns

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<th>Horizon h (in quarters)</th>
<th>pdt</th>
<th>cayt</th>
<th>R²(%)</th>
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<td>(-2.05)</td>
<td>(3.35)</td>
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<td></td>
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<td>(3.17)</td>
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<tr>
<td></td>
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<td>(4.81)</td>
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</table>

Panel B: Predictive regressions for bond returns

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<th>yt</th>
<th>CPt</th>
<th>R²(%)</th>
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Table 2 Table 3 Panel A reports monthly overlapping regressions of long horizon log stock returns onto a constant, the log price-dividend ratio and cayt. Table 3 Panel B reports monthly overlapping regressions of long horizon log total return on a 5-year constant maturity coupon bond from CRSP onto a constant, the log short rate y(t) and the yield spread spr(t). The short rate is the log yield on the 30-day Treasury Bill from CRSP, and the spread is the difference between the log yield on a 5-year artificial zero-coupon bond from the CRSP Fama-Bliss Discount Bond File, and the log yield on the Treasury Bill (T-bill). The table reports coefficient estimates, the $R^2$ of the regression, and, in brackets, the Newey-West corrected t-statistics.
<table>
<thead>
<tr>
<th></th>
<th>Long-Run Risk</th>
<th>External Habit</th>
<th>Rare Disaster</th>
<th>NA SDF Model</th>
<th>Lower Bound Data</th>
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<tr>
<td></td>
<td>Bansal, Bansal, Kiku-Yaron, Campbell, Bekaert-Engstrom</td>
<td></td>
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<td>std. dev.</td>
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<td>0.2996</td>
<td>0.3306</td>
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<tr>
<td>1-year mean</td>
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<td>5-year mean</td>
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<td>0.9277</td>
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<tr>
<td>std. dev.</td>
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<td>1.9498</td>
<td>1.5429</td>
<td>1.5133</td>
<td>1.0309</td>
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**Table 3 HJ Bounds with conditioning information.** We compute the variance of the SDFs at different holding horizons, via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. All calculation are based on model parameters tabulated in Favero, Ortu, Tamoni and Yang (2012, Table Appendix I - III), and the reported mean and standard deviation values from a single simulation run of 360,000 month. Then we construct the related 1-quarter, 1-year and 5-year SDFs. The reported lower bound is the minimum standard deviation of SDFs, based on real data from 1952:06 to 2010:09, at different horizons, using the optimally scaled bounds of Bekaert and Liu (2004), based on the return forecasting system (see (11)). Our assets set A contains market index, real 3-month T-bill returns, and 5-year maturity government bond. Assets set B contains market index, real 3-month Tbill returns, and 5-, 7-, 10, 20-, and 30-year maturity government bonds. The reported value in column NA SDF Model is the real data estimated valued according to a no-arbitrage SDF model proposed by Kojen, Lustig and Van Nieuwerburgh (2010) with sample size from 1952:06 to 2009:12. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.
We compute the variance of the SDFs at different holding horizons, via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. All calculation are based on model parameters tabulated in Favero, Ortu, Tamoni and Yang (2012, Table Appendix I - III), and the reported values are the average from a single simulation run of 360,000 month. Then we construct the related 1-quarter, 1-year and 5-year SDFs. We construct Hansen-Jagannathan Bounds at different horizons, using the optimally scaled bounds of Bekaert and Liu (2004), based on the return forecasting system (see (11)). Our assets set A contains market index, real 3-month T-bill returns, and 5-year maturity government. The quarterly data used in the construction of $\sigma$ is from 1952Q2 to 2010Q3, with the 90% confidence intervals. To compute the confidence intervals, we create 50,000 random samples of size 234 from the data, where the sampling in the block bootstrap is based on the optimal block length we calculated for each asset return regression residuals. The conservative $p$-values, shown in curly brackets for 1-quarter and 1-year horizon, represent the proportion of replications from which model-based $\sigma(m_t)$ exceeds lower bound of $\sigma_{\min}$’s 90% significance confidence intervals, in 200,000 replications of a finite sample simulation over 231 quarters. We set for the conservative $p$-values mean restrictions. It means $E(m_{1Y})$ is less than the threshold value in which case 90% simulated observations are considered. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.

<table>
<thead>
<tr>
<th></th>
<th>Long-Run Risk</th>
<th>External Habit</th>
<th>Rare Disaster</th>
<th>Lower Bound $\sigma_{min}$</th>
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<td>Cochrane-Bekaert-Engstrom</td>
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<td>1-quarter</td>
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<tr>
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<td>{0.060}</td>
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Table 4 HJ Bounds with conditioning information - SET A. We compute the variance of the SDFs at different holding horizons, via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. All calculation are based on model parameters tabulated in Favero, Ortu, Tamoni and Yang (2012, Table Appendix I - III), and the reported values are the average from a single simulation run of 360,000 month. Then we construct the related 1-quarter, 1-year and 5-year SDFs. We construct Hansen-Jagannathan Bounds at different horizons, using the optimally scaled bounds of Bekaert and Liu (2004), based on the return forecasting system (see (11)). Our assets set A contains market index, real 3-month T-bill returns, and 5-year maturity government. The quarterly data used in the construction of $\sigma$ is from 1952Q2 to 2010Q3, with the 90% confidence intervals. To compute the confidence intervals, we create 50,000 random samples of size 234 from the data, where the sampling in the block bootstrap is based on the optimal block length we calculated for each asset return regression residuals. The conservative $p$-values, shown in curly brackets for 1-quarter and 1-year horizon, represent the proportion of replications from which model-based $\sigma(m_t)$ exceeds lower bound of $\sigma_{\min}$’s 90% significance confidence intervals, in 200,000 replications of a finite sample simulation over 231 quarters. We set for the conservative $p$-values mean restrictions. It means $E(m_{1Y})$ is less than the threshold value in which case 90% simulated observations are considered. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.
Table 5 HJ Bounds with conditioning information - SET B. We compute the variance of the SDFs at different holding horizons, via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. All calculation are based on model parameters tabulated in Favero, Ortu, Tamoni and Yang (2012, Table Appendix I - III), and the reported values are the average from a single simulation run of 360,000 month. Then we construct the related 1-quarter, 1-year and 5-year SDFs. We construct Hansen-Jagannathan Bounds at different horizons, using the optimally scaled bounds of Bekaert and Liu (2004), based on the return forecasting system (see (11)). Our assets set B contains market index, real 3-month T-bill returns, and 5-, 7-, 10-, 20-, and 30-year maturity government bonds. The quarterly data used in the construction of the SDFs is from 1952Q2 to 2010Q3, with the 90% confidence intervals. To compute the confidence intervals, we create 50,000 random samples of size 234 from the data, where the sampling in the block bootstrap is based on the optimal block length we calculated for each asset return regression residuals. The conservative p-values, shown in curly brackets for 1-quarter and 1-year horizon, represent the proportion of replications from which model-based $\sigma(m_t)$ exceeds lower bound of $\sigma_{min}$’s 90% significance confidence intervals, in 200,000 replications of a finite sample simulation over 231 quarters. We set for the conservative p-values mean restrictions. It means $E(m_{1Y})$ is less than the threshold value in which case 90% simulated observations are considered. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.

<table>
<thead>
<tr>
<th></th>
<th>Long-Run Risk</th>
<th>External Habit</th>
<th>Rare Disaster</th>
<th>Lower Bound $\sigma_{\min}$</th>
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</thead>
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<tr>
<td></td>
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<td>Bansal, Yaron</td>
<td>Campbell, Cochrane</td>
<td>Bekaert, Engstrom</td>
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<tr>
<td>1-quarter</td>
<td>0.2871</td>
<td>0.3983</td>
<td>0.3019</td>
<td>0.2996</td>
</tr>
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<td></td>
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<td>{0.019}</td>
</tr>
<tr>
<td>5-year</td>
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<td>1.5133</td>
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<tr>
<td></td>
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</table>
Table 6 FS Bounds - SET A. We compute the variance of the SDFs at different holding horizons, via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. All calculation are based on model parameters tabulated in Favero, Ortu, Tamoni and Yang (2012, Table Appendix I - III), and the reported values are the average from a single simulation run of 360,000 month. Then we construct the related 1-quarter, 1-year and 5-year SDFs. We construct Hansen-Jagannathan Bounds at different horizon, using Ferson and Siegel (2003) approach, based on the return forecasting system (see (11)). Our assets set A contains market index, real 3-month T-bill returns, and 5-year maturity government. The quarterly data used in the construction of σ is from 1952Q2 to 2010Q3, with the 90% confidence intervals. To compute the confidence intervals, we create 50,000 random samples of size 234 from the data, where the sampling in the block bootstrap is based on the optimal block length we calculated for each asset return regression residuals. The conservative p-values, shown in curly brackets for 1-quarter and 1-year horizon, represent the proportion of replications from which model-based σ(mt) exceeds lower bound of σmin’s 90% significance confidence intervals, in 200,000 replications of a finite sample simulation over 231 quarters. We set for the conservative p-values mean restrictions. It means E(m1Y) is less than the threshold value in which case 90% simulated observations are considered. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.
<table>
<thead>
<tr>
<th></th>
<th>Long-Run Risk</th>
<th>External Habit</th>
<th>Rare Disaster</th>
<th>Lower Bound σ&lt;sub&gt;min&lt;/sub&gt;</th>
</tr>
</thead>
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<tr>
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<td>Bansal, Yaron</td>
<td>Bansal, Kiku-Yaron</td>
<td>Cochrane, Bekaert-Engstrom</td>
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<td>[2.066 3.170 ]</td>
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**Table 7 FS Bounds - SET B.** We compute the variance of the SDFs at different holding horizons, via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. All calculation are based on model parameters tabulated in Favero, Ortu, Tamoni and Yang (2012, Table Appendix I - III), and the reported values are the average from a single simulation run of 360,000 month. Then we construct the related 1-quarter, 1-year and 5-year SDFs. We construct Hansen-Jagannathan Bounds at different horizon, using Ferson and Siegel (2003) approach, based on the return forecasting system (see [11]). Our assets set B contains market index, real 3-month T-bill returns, and 5-, 7-, 10-, 20-, and 30-year maturity government bonds. The quarterly data used in the construction of σ is from 1952Q2 to 2010Q3, with the 90% confidence intervals. To compute the confidence intervals, we create 50,000 random samples of size 234 from the data, where the sampling in the block bootstrap is based on the optimal block length we calculated for each asset return regression residuals. The conservative p-values, shown in curly brackets for 1-quarter and 1-year horizon, represent the proportion of replications from which model-based σ(m<sub>t</sub>) exceeds lower bound of σ<sub>min</sub>’s 90% significance confidence intervals , in 200,000 replications of a finite sample simulation over 231 quarters. We set for the conservative p-values mean restrictions. It means E(m<sub>1Y</sub>) is less than the threshold value in which case 90% simulated observations are considered. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.
Table 8 Parameter Sensitivity Analysis of Bansal, Kiku and Yaron’s (2009) Model. We study the sensitivity of parameters, mainly about risk aversion and intertemporal elasticity substitution coefficients, for Bansal, Kiku and Yaron’s (2009) model. We construct Hansen-Jagannathan Bounds at different horizon, using Gallant, Hansen and Tauchen (1990) approach, based on the return forecasting system (see (11)). Our set A contains market index, real 3-month Tbill returns, and 5-year maturity government, and assets set B contains market index, real 3-month T-bill returns, and 5-, 7-, 10-, 20-, and 30-year maturity government bonds. The quarterly data used in the construction of $\sigma$ is from 1952Q2 to 2009Q4, with the 90% confidence intervals. To compute the confidence intervals, we create 50,000 random samples of size 234 from the data, where the sampling in the block bootstrap is based on the optimal block length we calculated for each asset return regression residuals. The conservative $p$-values, shown in curly brackets for 1-quarter and 1-year horizon, represent the proportion of replications from which model-based $\sigma(m_t)$ exceeds lower bound of $\sigma_{\text{min}}$’s 90% significance confidence intervals, in 200,000 replications of a finite sample simulation over 231 quarters. We set for the conservative $p$-values mean restrictions. It means $E(m_{1Y})$ is less than the threshold value in which case 90% simulated observations are considered. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.

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<td>{0.015}</td>
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<td>[2.409 2.434]</td>
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<td>[2.409 2.434]</td>
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<td>[2.409 2.434]</td>
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<tr>
<td><strong>Consumption growth volatility, $\sigma_t$</strong></td>
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<tr>
<td>Persistence</td>
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<tr>
<td>Volatility parameter</td>
<td>$\sigma_w$ $0.23 \times 10^{-5}$ $0.28 \times 10^{-5}$</td>
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</tr>
</tbody>
</table>

**Table 9 Appendix-I: Parametrization of asset pricing models incorporating long-run risk.** The long-run risk model of Bansal and Yaron (2004) and Bansal, Kiku and Yaron (2009) are based on Our parameterizations are based on Bansal and Yaron (2004, Table II) and Bansal, Kiku and Yaron (2009, Table I). The models are simulated at the monthly frequency.
<table>
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<th>Preferences</th>
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<th>Campbell-Cochrane</th>
<th>Bekaert-Engstrom</th>
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</thead>
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<td>Mean $\sigma$</td>
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<tr>
<td>Persistence</td>
<td>$\sigma_{xp} \times 10^3$</td>
<td></td>
<td>$-0.0016$</td>
</tr>
<tr>
<td>Volatility parameter</td>
<td>$\sigma_{xn} \times 10^3$</td>
<td></td>
<td>$-0.0140$</td>
</tr>
<tr>
<td>Volatility parameter $\sigma_{xp}$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Volatility parameter $\sigma_{xn}$</td>
<td>0.0052</td>
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<tr>
<td>Long-run risk</td>
<td>$p_t$ dynamics</td>
<td>$\bar{p}$</td>
<td>12.1244</td>
</tr>
<tr>
<td>Mean</td>
<td>$\rho_p$</td>
<td>0.9728</td>
<td></td>
</tr>
<tr>
<td>Persistence</td>
<td>$\sigma_{pp}$</td>
<td>0.4832</td>
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</tr>
<tr>
<td>Volatility parameter</td>
<td>$n_t$ dynamcis</td>
<td>$\pi$</td>
<td>0.6342</td>
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<tr>
<td>Mean</td>
<td>$\rho_n$</td>
<td>0.9810</td>
<td></td>
</tr>
<tr>
<td>Persistence</td>
<td>$\sigma_{nn}$</td>
<td>0.1818</td>
<td></td>
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<tr>
<td>Volatility parameter</td>
<td>Log of inverse consumption surplus, $q_t$</td>
<td>$\bar{q}$</td>
<td>1.0000</td>
</tr>
<tr>
<td>Mean</td>
<td>$\rho_q$</td>
<td>0.9841</td>
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</tr>
<tr>
<td>Persistence</td>
<td>$\sigma_{qp}$</td>
<td>0.0005</td>
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</tr>
<tr>
<td>Volatility parameter</td>
<td>$\sigma_{q}$</td>
<td>0.0479</td>
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</tr>
<tr>
<td>Steady state surplus consumption ratio $\bar{S}$</td>
<td>0.0570</td>
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<td></td>
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<tr>
<td>Persistence in consumption surplus ratio $\phi$</td>
<td>0.9884</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log of risk-free rate</td>
<td>$r^f \times 10^2$</td>
<td>0.0783</td>
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</tr>
</tbody>
</table>

Table 10 Appendix-II: Parametrization of asset pricing models incorporating external habit persistence. Our parameterizations are based on Campbell and Cochrane (1999) and Bekaert and Engstrom (2010). The models are simulated at the monthly frequency.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Annual</th>
<th>Monthly</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Preferences</strong></td>
<td></td>
<td></td>
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<tr>
<td>Time preference</td>
<td>δ</td>
<td>1.0448</td>
</tr>
<tr>
<td>Risk aversion</td>
<td>γ</td>
<td>5.190</td>
</tr>
</tbody>
</table>

**Gaussian component of consumption growth, $w_t$**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Annual</th>
<th>Monthly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>$\mu$</td>
<td>0.023</td>
</tr>
<tr>
<td>Volatility parameter</td>
<td>$\sigma$</td>
<td>0.01</td>
</tr>
</tbody>
</table>

**Non-gaussian component of consumption growth, $z_t$**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Annual</th>
<th>Monthly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean of Poisson density</td>
<td>$\omega$</td>
<td>0.010</td>
</tr>
<tr>
<td>Mean of $z_{t+1}$ conditional on $J_{t+1}$</td>
<td>$\theta$</td>
<td>$-0.30$</td>
</tr>
<tr>
<td>Variance of $z_{t+1}$ conditional on $J_{t+1}$</td>
<td>$\delta^2$</td>
<td>$0.15^2$</td>
</tr>
</tbody>
</table>

| Mapping between dividend and consumption, $D_t = C_t^\phi$ | $\phi$ | 2.60  | 2.60 |

**Table 11 Appendix-III: Parametrization of asset pricing models incorporating rare disasters.** Our parameterizations are based on Backus, Chernov, and Martin (2011, Table II, Column 2), annualized parameters. Then the monthly time preference parameter is the annual time preference parameter raised to the power $1/12$, see Campbell and Cochrane (1999). The leverage parameter $\lambda$ which follows Wachter (2011, Table 1).