Abstract

In this paper, I develop a model of self-reinforcing financial fads in which feedback patterns arise because of limitations to traders’ observational learning. I assume that the traders categorize price history into finite groups of price patterns due to their cognitive limitations, but their learning is rational subject to these constraints. Under asymmetric information about the fundamental value of the traded asset, the uninformed traders choose to follow the price trends as long as they perceive that the trends correctly reflect the true fundamentals. I find that abnormal price movements involving feedback loops endogenously emerge intermittently because the uninformed traders optimally follow even the abnormal price movements due to the coarseness of their learning. My findings have implications for bubble-like price patterns as well as momentum, reversal, and technical analysis.

JEL Classification Codes: G12, G14, D82, D83

Keywords: feedback effects, learning, bounded rationality, partitions, technical analysis, excessive extrapolation
1 Introduction

The self-reinforcing nature of price changes has often been considered one of the key factors that contribute to the occurrences of bubbles and crashes. A feedback loop of price changes can emerge due to wealth effects, consumption habits or credit expansions. However, it has been suggested that this can also emerge due to learning from prices. For example, according to Shiller (2002), “The essence of a speculative bubble is the familiar feedback pattern - from price increases to increased investor enthusiasm to increased demand and, hence, to further price increases.” Furthermore, feedback loops driven by observational learning may explain financial fads or manias, which occur in a short period, better than wealth effects, consumption habits, or credit expansions can. This is because other feedback mechanisms take a relatively longer time to build up and unwind. There have not been enough theoretical attempts to incorporate feedback loops of price changes driven by observational learning into financial equilibrium, despite its intuitive appeal and analytical usefulness.

In this paper, I attempt to explain why and how self-reinforcing fads can occur in financial equilibrium by analyzing a multiperiod trading model in which traders’ observational learning is constrained by partitions (or discrete categories). It is well known that trend-chasing behaviors can generate feedback patterns in financial markets (e.g., De Long, Shleifer, Summers, and Waldmann (1990b)). It has not, however, been fully explained why some traders would want to engage in the trend-chasing behaviors. I argue that the trend-chasing behaviors, which can lead to self-reinforcing fads, are consequences of natural responses to scarce cognitive resources rather than irrationality. Therefore, bubble-like price patterns that involve feedback loops can arise endogenously because of information processing constrained by categories.

Traders can have some limitations to their observational learning for many reasons such as scarce cognitive resources (e.g., attention and memory) and physical constraints (e.g., time). As a result, the traders who are constrained by scarce cognitive resources might have to use heuristic simplifications or a ‘rule of thumb’ in their observational inference. However, these heuristics are by no means irrational because they are based on sensible estimation procedures. I assume that the traders are boundedly rational in the sense that they know their limits to observational learning, and the learning is rational under such limitations. In particular, the traders categorize price history into finite groups of price patterns using partitions. Consequently, they only understand the average behavior of prices over the groups of price history, and choose to follow the price trends as long as they perceive that the trends correctly reflect the true fundamentals. Their observational learning is coarse in the sense that the extraction

---

1See, for example, Kindleberger (2000) for general surveys on this issue.
2See Kahneman (1973) for a more detailed discussion on the cognitive limits to the ability of reasoning.
3See Gilovich and Griffin (2002).
of private information from the price history is limited by the partitions. For example, a trader whose observational learning is constrained by the partitions will learn the same informational content from observing price changes that only differ by a few basis points. Nevertheless, the result of observational learning will be correct on average with any given partition.

In this paper, I focus on ‘fads’, which are bubble-like price patterns typically caused by the increasing optimism of a portion of investors. I define a bubble to be a situation in which prices deviate from the fundamental value of the traded asset for consecutive periods due to self-reinforcing belief updates. Suppose that the price goes up at time $t$. The traders at time $t+1$ who observe the previous return become more optimistic because they interpret the price movement as a good signal. As they increase their demand accordingly, the price at time $t+1$ goes up again. The upward price movement makes the traders at time $t+2$ more optimistic, so they increase their demand again. Consequently, the price at time $t+3$ goes up again, and induces more optimistic beliefs in the subsequent future, and so on. Therefore, price changes generate a self-feeding upward momentum through the channel of observational learning even though price changes in this case do not contain any informational content regarding fundamentals. I call this feedback loop of price changes that are purely driven by excessive extrapolation of past returns a ‘learning bubble’. It is a manifestation of self-reinforcing financial fads in which positive feedback trading is gradually amplified by its own effects. Figure 1 provides a schematic illustration of the feedback mechanism.

The possibility of a learning bubble is crucially dependent on whether the informed traders are equipped with finer observational partitions relative to the uninformed traders. In the case of a learning bubble, the competitive informed traders will prefer to ride predictable

---

4 Fads arise in the events in which some investors’ valuation becomes increasingly higher than the true value of the traded asset.

5 Bubbles could be defined in many different ways depending on the economic situations. Note that the bubble in this paper is different from the class of bubbles, which are often called ‘growth bubbles’, where the price is higher than the valuation of all agents (e.g., Tirole (1982) and Allen, Morris, and Postlewaite (1993)). See Camerer (1989) or Chapter 2 of Brunnermeier (2001) for more discussion on the definition of bubbles.
upward price movements despite the divergence of prices from fundamentals. Suppose that a simultaneous shift of the informed traders’ trading strategies is triggered by a particular path of the price history (or a ‘technical signal’) which predicts the occurrence of a learning bubble. If both the informed and uninformed traders are equipped with the same partitions, the uninformed traders also utilize the technical signal. Therefore, any upward price movement following the technical signal does not make the uninformed traders more optimistic about fundamentals. Thus, a learning bubble does not occur.

Now suppose that the informed traders’ observational partitions are finer than those of the uninformed traders. Even though the uninformed traders are not able to fully utilize each path of the price history for observational learning, they correctly weight the probability of the realization of each price path given the observed partition. If a technical signal realizes with a small probability, the arrival of the technical signal will trigger a simultaneous shift of the informed traders’ strategies without much impact on the belief updates of the uninformed traders. Unable to know the exact shift in the informed traders’ strategies, the uninformed traders excessively extrapolate past returns. Consequently, feedback loops of price changes emerge because the uninformed traders’ positive feedback trading gets amplified over time by its own feedback effects. In summary, learning bubbles can occur because the uninformed traders are unable to precisely distinguish between the price path that leads to a learning bubble and ones that do not. In this paper, I also show that crashes can follow bubbles through the same mechanism as those that create the learning bubbles.

Many financial anomalies can potentially be explained by excessive extrapolation of past returns. In this paper, I argue that such seemingly ‘excessive’ extrapolation might not be naive after all, and is rather a consequence of natural responses to scarce cognitive resources. Therefore, the equilibrium outcome driven by excessive extrapolation is stable even in the long run. In this paper, positive feedback trading is on average aligned towards the right direction. However, positive feedback trading can be intermittently amplified through feedback loops due to excessive extrapolation of past returns. This explains why positive feedback trading can cause momentum in general and why it does not necessarily cause reversal afterwards. That is, some incidents of momentum will always be followed by reversal because they are the consequences of feedback loops, which are driven by excessive extrapolation of past returns.

It is a puzzle why practitioners continue to rely on technical analysis even though academics have long been skeptical of it. However, recent empirical evidence such as Brock, Lakonishok,

---

6Strictly speaking, the traders observe an element (or price path) which belongs to a certain partition. In that case, the traders remember the partition rather than the element itself.

7See, for example, Lakonishok, Shleifer, and Vishny (1994) and Benartzi (2001) for evidence of excessive extrapolation of past returns.

8Momentum without reversal is the consequence of underreaction to observed information due to coarse observational learning.
and LeBaron (1992) and Lo, Mamaysky, and Wang (2000) finds a significant forecasting power in technical analysis. The results in this paper imply that technical signals can naturally emerge from the equilibrium over the space of observation, and can possess significant power for forecasting future prices due to their self-fulfilling nature. This is because technical signals do not necessarily deliver information regarding fundamentals, and in fact they deliver non-fundamental information such as the timing of rallies driven by some traders who understand the signals. This finding confirms Jegadeesh (2000) who describes “my discussions with practitioners suggested that chartists tend to use signals from patterns in the price history in conjunction with other information to time their trades.”

The paper is organized as follows: In Section 2 I review the related literature. In Section 3 I describe the investment opportunities, traders and information structure. In Section 4 I give the formal definition of the limits to observational learning and an equilibrium concept under such limits. In Section 5 I first solve a benchmark equilibrium when the informed traders are equipped with observational partitions as coarse as the uninformed traders. By introducing strictly finer observational partitions to the informed traders, I show that a learning bubble equilibrium emerges under certain conditions. In Section 6 I further extend the model to illustrate a learning bubble with a crash, and a slow-growing learning bubble. In Section 7 I discuss the empirical implications of the model regarding (i) bubbles and crashes, (ii) price behaviors such as momentum, reversal and excess volatility, and (iii) technical analysis.

2 Related Literature

This paper is related to four strands of literature. The first strand studies the feedback effects of price changes. Most of the literature in this category focuses on the interaction between price changes and their impact on fundamentals (e.g., Dow and Rahi (2003), Hirshleifer, Subrahmanyam, and Titman (2006) and Ozdenoren and Yuan (2007)). This paper is closest to De Long, Shleifer, Summers, and Waldmann (1990b) who focus on the feedback effects of price changes caused by positive feedback traders. De Long, Shleifer, Summers, and Waldmann (1990b) show that informed speculators can jump onto rising prices when there are positive feedback traders. In this paper, I extend the results of De Long, Shleifer, Summers, and Waldmann (1990b) in the following directions: First, I demonstrate that positive feedback trading can be generated by a belief system that is consistent with equilibrium while De Long, Shleifer, Summers, and Waldmann (1990b) are silent about why positive feedback trading arises.

Second, I focus on why and when positive feedback trading gets amplified by feedback loops of price changes. Third, I study the impact of positive feedback trading over a longer horizon.

Consistent belief means the belief system is on average correct in equilibrium.
enabling the model to explain more empirical implications which include momentum, reversal and excess volatility. In particular, the findings of this paper explain why reversal might or might not always follow momentum.

The second strand of literature studies how financial fads can arise due to observational learning. For example, Avery and Zemsky (1998) find that some specific structures of signals can lead to information cascades. This strand typically studies how information cascades arise in sequential trading models, as shown in Glosten and Milgrom (1985), as a result of herd behavior driven by observational learning (e.g., Bikchandani, Hirshleifer, and Welch (1992), Lee (1998), Avery and Zemsky (1998), Decamps and Lovo (2006) and Park and Sabourian (2008)). Due to the existence of risk-neutral market makers with infinite liquidity, the price is usually set to be the market makers’ conditional expectation of the fundamental value of an asset. Herd behavior of informed traders typically leads to information cascades. Consequently, prices that are equal to the valuation of the market makers do not change any more once herd behavior starts. Although the sequential trading setup successfully highlights why herd behavior can occur due to the failure of observational learning, it does not explain how herd behavior might lead to further abnormal price behavior afterwards. On the contrary, I study trading behavior driven by observational learning in a competitive market setup. One advantage to adopting this setup is that the price is set by a Walrasian auctioneer at a market clearing price. Therefore, the price can still respond to increased demand even after the herd behavior of informed traders. It enables me to study the feedback effects of price changes, which can be an essential feature of the bubbles.

The third strand of literature explains financial anomalies such as bubbles, momentum, and excess volatilities by adopting behavioral or psychological biases (e.g., De Long, Shleifer, Summers, and Waldmann (1990a), Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subramanyam (1998), Odean (1998) and Scheinkman and Xiong (2003)). For example, De Long, Shleifer, Summers, and Waldmann (1990a) shows that the existence of investor sentiment leads to mispricing due to limited arbitrage. Scheinkman and Xiong (2003) show that overconfident traders with heterogeneous belief can get into speculative trades, thereby creating bubbles. The literature in this strand is usually silent about whether those belief systems would survive in the long run. This paper complements this strand of literature by showing why there can exist subjective beliefs departing from objective beliefs by endogenizing the belief system into the model under a bounded rationality framework. Furthermore, this strand of literature usually assumes that information is exogenously given rather than endogenously learned from prices. This paper contributes to an emerging literature by adding a new mechanism of generating financial instability through observational learning.

The fourth strand of literature focuses on how bubbles arise or are sustained due to the
coordination failure of competitive informed traders (e.g., De Long, Shleifer, Summers, and Waldmann (1990a), Shleifer and Vishny (1997) and Abreu and Brunnermeier (2003)). For example, Abreu and Brunnermeier (2003) show that a bubble can persist because informed traders are unable to burst it due to the dispersion of opinions and the lack of temporary coordination. This paper complements this line of literature as follows: First, I track the formation and collapse of bubbles, which are triggered by the simultaneous actions of the informed traders. Second, the coordination mechanism considered in this paper is given by technical signals that emerge in the price history. Third, this paper contributes to an emerging literature in this strand by linking bubble phenomena with observational learning from prices.

In canonical noisy rational expectation equilibrium (NREE) models such as Grossman and Stiglitz (1980), Wang (1993) and Wang (1994), the price can depart from the fundamentals due to adverse liquidity shocks. In the case where similar kinds of shocks arrive consecutively, mispricing can become larger and the uninformed traders’ beliefs can further deviate from the fundamentals. However, the feedback loops of price changes during a learning bubble fundamentally differ from the mispricing caused by a series of adverse liquidity shocks because a liquidity shock does not create any feedback effects by itself. That is, the initial mispricing is not magnified in the subsequent period unless similar kinds of liquidity shocks happen to arrive again. Furthermore, in this model, the feedback loops arise because the occurrence of a learning bubble is based on private information as well as the fundamental value of the risky asset. The information regarding the occurrence of the feedback loops is only known to the speculators who have superior ability to interpret the price history. Although canonical NREE models only consider information asymmetries on fundamentals, I add an extra dimension of information asymmetries on non-fundamental information about the shift of other informed traders’ strategies.

Also, there is a growing literature that studies various economic phenomena in case agents’ information processing is constrained by categories or groups (e.g., Mullainathan (2002), Jehiel (2005), Fryer and Jackson (2008) and Mullainathan, Schwartzstein, and Shleifer (2008)). The equilibrium concept in this paper is related to Analogy-Based Expectations Equilibrium introduced in Jehiel (2005), who studies a game theoretic equilibrium with players who only understand the average behavior of their opponent over bundles of states. Players in the game defined in Jehiel (2005) use analogy-based inference on other traders’ equilibrium behavioral strategy where players bundle states into analogy classes, and play best-response to their opponent’s average strategy in those analogy classes. In this paper, traders perform observational inference on equilibrium prices instead of other players’ actions. Furthermore, the traders in this model do not have to know other traders’ strategies themselves as long as they are aware of the equilibrium behavior of prices. In their observational learning, the traders bundle

\[10\] In the case of a large population game such a trading game in financial market among infinitely many
observed price changes into disjoint partitions (or analogy classes in terms of Jehiel (2005)) and forms their average belief given the observation.

3 Basic Model

I consider a multiperiod trading model in a finite horizon where trading occurs at each time $t \in T \equiv \{0, 1, \ldots, T\}$. There exists a riskless asset yielding a fixed return in every period with perfectly elastic supply. I normalize the gross return of the riskless asset to one for simplicity. There also exists a risky asset that yields the liquidation value $V$ at time $T + 1$. There are two states of the world where the state space is given by $\Omega = \{G, B\}$. Each unit of the share of the risky asset yields the payoff of $V = 1$ in the good state $\omega = G$ and $V = 0$ in the bad state $\omega = B$. The prior distribution of the state of the world, which is common knowledge among all traders, is given by

$$
\omega \in \begin{cases} 
G, & \text{with probability } \pi; \\
B, & \text{with probability } 1 - \pi;
\end{cases}
$$

where $0 < \pi < 1$.

All traders are infinitesimal. Thus, they are all price takers. There are two types of traders in the economy: uninformed long-term traders (denoted by $U$) and informed speculators (denoted by $I$). The long-term traders are long-lived, risk-averse, and uninformed about the true state of the world. They have an identical CARA utility function with a risk aversion parameter $\gamma$. They also have an identical initial wealth $W_0$ at time $t = 0$. I assume that they form their demand based only on the price relative to the fundamental value of the risky asset similarly to ‘passive investors’ in De Long, Shleifer, Summers, and Waldmann (1990b) or ‘newswatchers’ in Hong and Stein (1999). Thus, they only consider the long-term prospect of the risky asset rather than the short-term speculative gains. On the other hand, the speculators are short-lived, risk-neutral, and perfectly informed about the true state of the world. They have no initial wealth, and trade on margin to finance their investment. For simplicity, I assume that the margin constraint is given as a form of the maximum number of long or short positions, i.e., $x_t \in [-M, M]$ for all $t = 1, 2, \ldots, T - 1$ where $x_t$ is the position held by a trader. It is also difficult for each trader to follow every action of other traders in the market.

11 Although the long-term traders trade dynamically, their demand is based on the static-optimization. That is, the long-term traders have time-inconsistency in their dynamic portfolio choice as the passive investors in De Long, Shleifer, Summers, and Waldmann (1990b) or the newswatchers in Hong and Stein (1999). The role of the time-inconsistent investors in the model is to set the price towards their valuation of the risky asset including the risk premium regardless of the speculative gains. Therefore, they work as a stabilizing force in the model as long as their belief is well aligned with the fundamentals.
The long-term traders arrive at the market at time $t = 0$ and trade until the liquidation at time $t = T + 1$. On the other hand, a new group of speculators arrives at the market at each time $t = 1, 2, \ldots, T - 1$, and leaves the market in the subsequent period. I normalize the mass of the long-term traders and the new group of speculators who arrive at the market at each trading time $t = 1, 2, \ldots, T - 1$ to be one, respectively. The timing of the model is summarized in Figure 2.

I assume that the net supply of shares $y_t$ is initially set to zero (i.e., $y_0 = 0$), and follows an autoregressive process:

$$y_t = y_{t-1} + \epsilon_t,$$

where $\epsilon_t$ is a new innovation to the process of $y_t$. The random variable $\epsilon_t$ is independent over time, and has a realization such that

$$\epsilon_t \in \begin{cases} 
\bar{y} + \rho, & \text{with probability } \eta; \\
\bar{y}, & \text{with probability } \frac{1-\theta}{2} - \eta; \\
0, & \text{with probability } \theta; \\
-\bar{y}, & \text{with probability } \frac{1-\theta}{2} - \eta; \\
-\bar{y} - \rho, & \text{with probability } \eta;
\end{cases}$$

where $\eta, \rho$ are non-negative numbers. I index the innovation to the supply of shares by ‘more positive’ (++), ‘positive’ (+), ‘null’ (Ø), ‘negative’ (−), and ‘more negative’ (−−) for $\bar{y} + 12$

This is a very simplified assumption regarding the margin requirement. See Shleifer and Vishny (1997), Gromb and Vayanos (2002), Attari, Mello, and Ruckes (2005) and Brunnermeier and Pedersen (2009) for the more specialized setup.
\( \rho, \bar{y}, 0, -\bar{y}, \) and \( -\bar{y} - \rho, \) respectively. I assume that the magnitude of the innovation to the supply of shares is always smaller than the maximum aggregate demand of the speculators, i.e., \( 0 < \bar{y} + \rho < M. \) I further assume that \( \rho \) is an increasing function of \( \eta, \) e.g., \( \rho(\eta) = \eta. \) Therefore, one could interpret the innovation ++ (or --) as a small deviation from + (or −) when \( \eta \) is small.

4 Equilibrium Concept under Coarse Observational Learning

4.1 Coarse Observational Learning

I assume that the traders receive the data on the price history in the form of returns\(^{13}\) Let \( \mathcal{R}_t = \mathbb{R} \) be the space of returns at time \( t. \) Then, the space of the price history at time \( t \) is given by a product of the spaces of returns until time \( t: \Lambda_t \equiv \mathcal{R}_1 \times \mathcal{R}_2 \times \ldots \times \mathcal{R}_t. \) For example, the traders at time \( t \) will receive the data on the price history such that \( \lambda_t \in \Lambda_t \) where \( \lambda_t \equiv (r_1, r_2, \ldots, r_t) \) and \( r_z \in \mathcal{R}_z \) are the realized return at each time \( z = 1, 2, \ldots, t. \) The long-term traders attempt to infer the true state of the world from the observation of \( \lambda_t. \) Therefore, the observational learning of the long-term traders at time \( t \) can be represented by the following mapping\(^{14}\)

\[
\mu^U_t (\cdot | \lambda_t \in \Lambda_t) : \Omega_t \rightarrow [0, 1].
\] (1)

On the other hand, the speculators attempt to predict the price in the subsequent future conditional on \( \omega \) and the observation of \( \lambda_t. \) Therefore, observational learning of the speculators at time \( t \) can be represented by the following mapping\(^{15}\)

\[
\mu^I_t (\cdot | \lambda_t \in \Lambda_t, \omega \in \Omega) : \mathcal{P}_{t+1} \rightarrow [0, 1],
\] (2)

where \( \mathcal{P}_{t+1} \) is the set of feasible prices at time \( t + 1. \)

\(^{13}\)The form of the reported return can be any form such as the logarithmic or arithmetic return or the difference of prices. For example, suppose \( p_{t-1}, p_t \) are the prices at time \( t-1 \) and time \( t, \) respectively. Then, the traders at time \( t \) can observe the return data \( r_t \equiv \log \left( \frac{p_t}{p_{t-1}} \right). \)

\(^{14}\)For example, if a long-term trader assigns 0.8 to the posterior probability of \( \omega = G \) given \( (r_1, r_2), \) I say \( \mu^U_t(G|(r_1, r_2)) = 0.8. \)

\(^{15}\)For example, if a speculator assigns 0.2 to the posterior probability of \( p_{t+1} = 0.5 \) given \( (r_1, r_2) \) and \( \omega = G, \) then I say \( \mu^I_t(0.5|(r_1, r_2), G) = 0.2. \)
ring information on a partitioned signal space when the state space is continuous. Another example is the perception of prices that are constrained by decimal numbers. This perception can contribute to the well-documented empirical phenomena of price clustering or resistance points of stock prices around decimal levels (e.g., Niederhoffer (1965), Niederhoffer (1966), Harris (1991), Donaldson and Kim (1993) and Sonnemans (2006)). Such partitions, which compress the amount of information, can be considered as heuristic simplifications which are natural responses to cognitive constraints.

In this paper, I assume that the traders are equipped with discrete partitions that constrain their observational learning. I give a formal definition of coarse observational partitions as follows:

**Definition 1** The *return partition* of trader $a$ at time $t$ is a member of $\mathcal{R}_t^a \equiv \{ R_{j,t}^a \}_{j=1}^N$, which is a collection of intervals such that $R_{j,t}^a = (-\infty, r_1), R_{2,t}^a = [r_1, r_2), \ldots, R_{N,t}^a = [r_N, \infty)$ for some sequence $-\infty < r_1 < r_2 < \ldots < r_N < \infty$.

**Definition 2** The *history partition* of trader $a$ at time $t$ is a member of $\mathcal{J}_t^a \equiv \{ J_{j,t}^a \}_{j=1}^N$, which is a collection of non-empty subsets of $\Lambda_t$ such that

(i) $J_{j,t}^a \equiv R_{j,1}^a \times R_{j,2}^a \times \ldots \times R_{j,t}^a$ where $R_{j,z}^a \subset \mathbb{R}$ for all $z = 1, 2, \ldots, t$,

(ii) $J_{j,t}^a$’s are mutually disjoint,

(iii) $\cup_{j=1}^N J_{j,t}^a = \Lambda_t$.

For notational convenience, I denote $J_{j,t}^a(\lambda_t)$ to be the history partition of trader $a$ at time $t$ that includes $\lambda_t$. I also define coarse observational learning that is constrained by $\mathcal{J}_t^U, \mathcal{J}_t^I$, respectively:

**Definition 3** (i) The long-term traders at time $t$ have **coarse observational learning** if $\mu_t^U(\omega|\lambda_t) = \mu_t^U(\omega|\lambda'_t)$ for all $\omega \in \Omega$ and all $\lambda_t, \lambda'_t \in J_{j,t}^U$ given any $J_{j,t}^U \in \mathcal{J}_t^U$.

(ii) The speculators at time $t$ have **coarse observational learning** if $\mu_t^I(p_{t+1}|\lambda_t, \omega) = \mu_t^I(p_{t+1}|\lambda'_t, \omega)$ for all $p_{t+1} \in \mathcal{P}_{t+1}, \omega \in \Omega$ and all $\lambda_t, \lambda'_t \in J_{j,t}^I$ given any $J_{j,t}^I \in \mathcal{J}_t^I$.

Therefore, a trader who performs coarse observational learning deduce the same information content from two different price paths if they belong to the same history partition. Coarse observational learning is consistent with the equilibrium if and only if the learning is correct on average in each history partition.

**Definition 4** (i) $\mu_t^U$ is **consistent** with the history $\lambda_t$, if

$$\mu_t^U(\omega|\lambda_t) \equiv \sum_{\lambda'_t \in J_{j,t}^U(\lambda_t)} \frac{Pr(\lambda'_t) Pr(\omega|\lambda'_t)}{\sum_{\lambda''_t \in J_{j,t}^U(\lambda_t)} Pr(\lambda''_t)}$$

(3)

---

16 See Hirshleifer (2001) for more discussion on heuristic simplifications.
(ii) $\mu^I_t$ is **consistent** with the history $\lambda_t$, if

$$
\mu^I_t(p_{t+1} | \lambda_t, \omega) \equiv \sum_{\lambda'_t \in \mathcal{H}^I(\lambda_t)} \frac{Pr(\lambda'_t | \omega) Pr(p_{t+1} | \lambda'_t, \omega)}{\sum_{\lambda'_t \in \mathcal{H}^I(\lambda_t)} Pr(\lambda'_t | \omega)}.  \tag{4}
$$

A trader with consistent coarse observational learning puts the correct weighting on each possible price path in the given history partition. This condition is equivalent to the ‘weak consistency’ in Jehiel (2005). Similar to Jehiel (2005), the interpretation of this condition goes as follows: If a trader repeatedly participates in trading, then the trader will eventually understand the correct weighting that coincides with the true frequency of each price path in the given history partition.

### 4.2 The Evolution of the Long-term Traders’ Beliefs

For simplicity, I call $p_t \geq p_{t-1}$ an ‘upward’ price movement, and $p_t < p_{t-1}$ a ‘downward’ price movement. I assume that the long-term traders’ return partitions at time $t$ are given by

$$
\mathcal{R}^U_z = \{(-\infty, 0), [0, \infty)\} \text{ for } z = 1, 2, \ldots, t - 1,
$$

and

$$
\mathcal{R}^U_t = \{(-\infty, \infty)\}.
$$

Therefore, the long-term traders’ observational learning is ‘binary’ and ‘delayed’. That is, the long-term traders only learn from either upward or downward price movements that are in the past. Since their learning is delayed, they do not infer the state of the world from the Walrasian auctioneer’s price offers at the current period. A similar delay in observational learning in a Walrasian auction setup has been considered in Hellwig (1982). On the contrary, I assume that the speculators’ observational learning is not delayed. I say that the speculators’ observational partitions are ‘finer than’ those of the long-term traders if they have finer return partitions than binary partitions in every period. I say that the speculators’ observational partitions are ‘as coarse as’ those of the long-term traders if the speculators also have binary return partitions. For example, the long-term traders in the current period and the speculators in the previous period will observe the same signal from the price history if the speculators’ observational partitions are as coarse as those of the long-term traders.

---

17Note that their demand is still elastic to price changes regardless of the coarseness of their observational partitions.

18There are also other reasons to create delayed response to information; for example, Vayanos and Woolley (2008) considers gradual adjustment of the demand to observed information due to contractual restrictions or institutional lags in a continuous time framework.
The set of observable price changes at time $t$ can be represented by $S_t \equiv \{u_t, d_t\}$ where $u_t, d_t$ denotes the upward and downward price movement at time $t$, respectively. I index the price change at time $t$ as $s_t \in S_t$. For notational convenience, I will denote the long-term traders’ history partition at time $t$ by suppressing the superscript $U$ as follows:

$$J_{t-1}(\lambda_t) \equiv J^U_t(\lambda_t) = \{s_1, s_2, \ldots, s_{t-1}\}.$$  

The long-term traders’ beliefs about $\omega = G$ conditional on $\lambda_t$, of which derivation is relegated to Appendix A, is given by:

$$
\mu^U_t(\omega|\lambda_t) = \frac{Pr(J_{t-1}(\lambda_t)|G)\pi}{Pr(J_{t-1}(\lambda_t)|G)\pi + Pr(J_{t-1}(\lambda_t)|B)(1-\pi)}, \tag{5}
$$

I define $l_t$ to be the log-likelihood ratio of the long-term traders’ beliefs at time $t$, which measures the odds of being in the state $G$ versus $B$ given the long-term traders’ beliefs based on the observed history $\lambda_t$:

$$l_t \equiv \log \left( \frac{\mu^U_t(G|\lambda_t)}{\mu^U_t(B|\lambda_t)} \right) = \log \left( \frac{Pr(G|J_{t-1}(\lambda_t))}{Pr(B|J_{t-1}(\lambda_t))} \right).$$

Then, the initial condition is given by $l_1 = l_0 = \log \left( \frac{\pi}{1-\pi} \right)$ due to delayed observational learning. The following lemma is derived from (5) and the definition of $l_t$:

**Lemma 1** *The evolution of the long-term traders’ beliefs about $\omega$ is governed by the following linear equation with the initial condition $l_1 = \log \left( \frac{\pi}{1-\pi} \right)$:*

$$l_t = l_{t-1} + \Delta l_{s,t},$$

where

$$\Delta l_{s,t} = \begin{cases} 0, & \text{if } Pr(s_{t-1}|J_{t-2}(\lambda_{t-1}), G) = Pr(s_{t-1}|J_{t-2}(\lambda_{t-1}), B) = 0; \\ \log \left( \frac{Pr(s_{t-1}|J_{t-2}(\lambda_{t-1}), G)}{Pr(s_{t-1}|J_{t-2}(\lambda_{t-1}), B)} \right), & \text{otherwise}; \end{cases}$$
4.3 Coarse Learning Expectations Equilibrium

I denote the conditional expectation of the long-term traders and the speculators at time \( t \) under coarse observational learning using a superscript \( U \) and \( I \), respectively:

\[
E_t^U \left[ U \left( W_0 + (V - p_t)x \right) \right] = - \sum_{\omega' \in \Omega} \mu_t^U (\omega' | \lambda_t) \exp \left( -\gamma \left( W_0 + (V(\omega') - p_t)x \right) \right); \\
E_t^I \left[ (p_{t+1} - p_t)x \right] = \sum_{p'_{t+1} \in P_{t+1}} \mu_t^I (p'_{t+1} | \lambda_t, \omega) (p'_{t+1} - p_t)x,
\]

where \( V(\omega) \) is the liquidation value of the risky asset in the state \( \omega \). Also, a symmetric equilibrium under coarse observational learning is formalized as follows.

**Definition 5** The sequence \((p_t, x_t^U, x_t^I)_{t=0}^T\) is a Coarse Learning Expectations Equilibrium (CLEE) under the observational partitions \( J_t^U, J_t^I \) if there exist beliefs \( \mu_t^U, \mu_t^I \) such that

(i) \( x_t^U \in \arg\max_{x \in (-\infty, \infty)} E_t^U \left[ U \left( W_0 + (V - p_t)x \right) \right] \) for all \( t = 0, 1, \ldots, T \),

(ii) \( x_t^I \in \arg\max_{x \in [-M, M]} E_t^I \left[ (p_{t+1} - p_t)x \right] \) for all \( t = 1, 2, \ldots, T-1 \) and \( x_T^I = 0 \) for all \( t = 0, T \),

(iii) \( \mu_t^U, \mu_t^I \) are consistent with \( \lambda_t \),

(iv) \( x_t^U + x_t^I = y_t \) for all \( t = 0, 1, \ldots, T \).

4.4 Learning Bubble Equilibrium

I call a feedback loop of upward price changes that is driven by excessive extrapolation of past returns a ‘learning bubble’. A learning bubble equilibrium is a CLEE in which a learning bubble occurs with a positive probability in the bad state. There can exist other types of equilibria in which a feedback loop of downward price changes occur in the good state. For simplicity of analysis, I only consider the CLEE with a feedback loop of upward price changes in the bad state.

I formalize the concept of a learning bubble equilibrium. First, I define \( \Delta \bar{l} \) to be an endogenous parameter that does not depend on time or sample path. \( \bar{l} \) is an endogenous parameter because it is computed in the equilibrium.

**Definition 6** A CLEE is a learning bubble equilibrium if a learning bubble arises with a positive probability in the bad state, i.e., there exists \( T^* \subset T \) that satisfies the followings when \( \omega = B, \{\epsilon_t\}_{t=1}^T \in \hat{\Sigma} \) with \( Pr(\omega = B, \{\epsilon_t\}_{t=1}^T \in \hat{\Sigma}) > 0 \):

\[19\] Since the expected utilities given all observations within the same partition are identical, the set of optimal strategies is also identical. I focus on the case where the equilibrium strategy is identical for each type.

\[20\] \( \Delta \bar{l} \) is an endogenous parameter because it is computed in the equilibrium.
(i) a positive return in the previous period implies more optimistic beliefs of the long-term traders in the current period, during $T^*$, i.e.,

$$\Delta l_{u,t} > \Delta \bar{l} \text{ if } p_{t-1} > p_{t-2} \text{ for all } t \in T^*,$$

(ii) more optimistic beliefs of the long-term traders in the current period imply a positive return in the current period, during $T^*$, i.e.,

$$p_t > p_{t-1} \text{ if } \Delta l_{u,t} > \Delta \bar{l} \text{ for all } t \in T^*.$$

5 Equilibrium

5.1 The Benchmark Case

In this section, I solve the equilibrium of the model in which the speculators’ observational partitions are as coarse as those of the long-term traders. For this benchmark case, I assume that $\eta = 0$, i.e., each $\epsilon_t$ has only three possible states: $\{-, \emptyset, +\}$. I give the results directly by relegating the derivation of them to Appendix B. The aggregate demand of the long-traders is given by

$$x^{U}_t = \frac{1}{\gamma} \left[ l_t + \log \left( \frac{1 - p_t}{p_t} \right) \right], \quad (6)$$

while the aggregate demand of the speculators is given by

$$x^I_t \in \begin{cases} 
M, & \text{if } E^I_t[p_{t+1}] > p_t; \\
[-M, M], & \text{if } E^I_t[p_{t+1}] = p_t; \\
-M, & \text{if } E^I_t[p_{t+1}] < p_t. 
\end{cases} \quad (7)$$

Solving the market clearing condition, the equilibrium price at time $t$ given $l_t, y_t$ and $x^I_t$ is represented as\footnote{\text{The market clearing condition is given by $x^I_t + x^{U}_t = y_t$.}}

$$p_t = \frac{1}{1 + \exp \left( -l_t + \gamma(y_t - x^I_t) \right)}. \quad (8)$$

The following lemma is immediately derived from the above result:
Lemma 2 The return at time $t$ has the same sign as $\Delta l_{s,t} - \gamma [\epsilon_t - (x^I_t - x^I_{t-1})]$, i.e.,

$$\text{sign}(p_t - p_{t-1}) = \text{sign}(\Delta l_{s,t} - \gamma [\epsilon_t - (x^I_t - x^I_{t-1})]).$$

5.1.1 The Existence of Equilibrium

There always exists a fully-revealing CLEE such that the price converges to the true value at time $t = 2$. Under certain conditions, there exists an equilibrium that partially reveals the private information of the informed speculators at every period. I focus on the partially-revealing equilibrium rather than other types of equilibria. I directly give the result in the following proposition by relegating the proof to Appendix C:

Proposition 1 (Partially-revealing Equilibrium) There exist a partially-revealing CLEE under some parameter values of $M$ and $\theta$ such that the following properties are true for $t = 1, 2, \ldots, T - 1$:

1. **State-dependent Return Process**: (i) When $\omega = G$, $p_t \geq p_{t-1}$ if $\epsilon_t \in \{-, \emptyset\}$, and $p_t < p_{t-1}$ if $\epsilon_t \in \{+\}$, (ii) When $\omega = B$, $p_t \geq p_{t-1}$ if $\epsilon_t \in \{-\}$, and $p_t < p_{t-1}$ if $\epsilon_t \in \{\emptyset, +\}$.

2. **Momentum of Expected Return**: For all $\omega \in \{G, B\}$, $E^I_t[p_{t+1}] \geq p_t$ if $p_t \geq p_{t-1}$, and $E^I_t[p_{t+1}] < p_t$ if $p_t < p_{t-1}$.

The State-dependent Return Process Property (Property-S) states that the return in the current period depends on the true state as well as the innovation to the supply of shares. Furthermore, the price is more likely to go up in the good state, and is more likely to go down in the bad state. The Momentum of Expected Return Property (Property-M) states that the return at time $t + 1$ is expected to continue in the same direction as the return at time $t$.

Figure 3 shows simulated examples of partially-revealing CLEE that satisfy both Property-S and Property-M.

The evolution of the long-term traders’ beliefs under Property-S is given by the following linear equation with the initial condition $l_1 = \log(\pi/(1 - \pi))$:

$$l_t = l_{t-1} + \Delta l_{s,t},$$

The proof is presented in Appendix C.

Because the direction of the return at time $t$ is determined only by $\omega$ and $\epsilon_t$, it is independent of the previous returns.
Figure 3. Simulated examples of partially-revealing equilibrium when $\omega = G$ (left) and $\omega = B$ (right) (Parameter values: $T = 500, \theta = 0.08, M = 0.1, \bar{y} = 0.01, \gamma = 1, \pi = \frac{1}{2}$)

where

$$
\Delta l_{s,t} = \begin{cases} 
\log\left(\frac{1+\theta}{1-\theta}\right), & \text{if } s_{t-1} = u_{t-1}; \\
-\log\left(\frac{1+\theta}{1-\theta}\right), & \text{otherwise};
\end{cases}
$$

for all $t = 2, 3, \ldots, T$. The change in the long-term traders’ beliefs at time $t$ is either positive or negative depending on the sign of $s_{t-1}$. Therefore, the long-term traders always engage in positive feedback trading because of their delayed learning. Property-M shows that prices are expected to have momentum given the information set of the speculators. This is because of the long-term traders’ delayed response to observed information as well as the portfolio constraints of the speculators. The aggregate demand of the speculators is given by a ‘bang-bang’ solution due to (7) and Property-M for all $t = 1, 2, \ldots, T - 1$:

$$
x^I_t = \begin{cases} 
M, & \text{if } p_t \geq p_{t-1}; \\
-M, & \text{otherwise}.
\end{cases}
$$

Finally, the proof in the appendix shows the existence of some parameter regions that satisfy Property-S and Property-M given the optimal demand of the traders. It is worth mentioning about the parameter regions that support the partially-revealing equilibrium. If prices change too much in one period, the curvature of the price function (8) becomes too significant relative to main factors such as $l_t, x^I_t$ and $y_t$. On the other hand, small enough price changes minimize the effect of the curvature of the price function. Therefore, the price change in each period needs to be small enough for the equilibrium to exist. The proof in the appendix shows that such a condition is achieved when $M$ is small enough. That is, the volume of informed trading
needs to be smaller than a threshold that does not fully reveal the private information of the informed traders.\footnote{In case informed trading volume is large enough, there only exists fully-revealing equilibrium.}

### 5.1.2 The Non-existence of a Learning Bubble Equilibrium

In this section, I show that there exists no learning bubble equilibrium if the speculators’ observational partitions are as coarse as those of the long-term traders. The result in the previous section shows that the long-term traders always engage in positive feedback trading because of their delayed response to price changes. However, the long-term traders will never engage in positive feedback trading in the future if they realize that price changes are solely created by the feedback effects of their own trading volume. As long as the uninformed traders can figure out the shift of the informed traders’ trading strategies, a learning bubble, which is a feedback loop of price changes driven by excessive extrapolation of past returns, cannot occur. The following lemma shows that the two conditions in Definition \ref{def:learning-bubble} cannot be true together when the speculators’ observational partitions are as coarse as those of the long-term traders.

**Proposition 2 (The Impossibility of a Learning Bubble)** Theorem exists no CLEE that is a learning bubble equilibrium if the informed speculators’ observational partitions are as coarse as those of the uninformed long-term traders.

**Proof** See Appendix D.

Therefore, there will not be any feedback effect of price changes driven by observational learning as long as the uninformed traders can precisely detect potential triggers of feedback loops. Proposition 2 implies that there cannot be any feedback effects of price changes in canonical NREE models because agents in those models have the same ability in observational learning. That is, there is no extra signal that coordinates a simultaneous shift of informed agents’ trading strategies without the knowledge of uninformed agents in those models.\footnote{Prices can depart from the fundamental value due to a series of adverse liquidity shocks in canonical NREE models, but the liquidity shocks by themselves do not create any feedback effect.}

### 5.2 Learning Bubbles

In this section, I show that learning bubbles can arise if the uninformed traders perform coarser observational learning than the informed traders. I assume that the speculators have fine enough return partitions which enable them to infer the realizations of $\epsilon_t$ precisely from price history.\footnote{Note that $\epsilon_t$ is the only source of randomness in this model. That is, there is only a finite number of feasible price paths because $\epsilon_t$ has only a finite number of realizations. Therefore, there always exists a set of fine} I further assume that $\eta$ is strictly positive.
I briefly describe the general idea of the proof here: Suppose that a certain path of price history conveys a technical signal that predicts a series of upward price movements in the subsequent future. Also suppose that the speculators’ observational partitions are finer than those of the long-term traders, and the technical signal is precisely understood only by the speculators. The speculators can simultaneously shift their trading strategies depending on the arrival of the technical signal while the uninformed traders only recognize the possibility of the arrival of the technical signal. Unable to know the exact shift in the informed traders’ strategies, the uninformed traders update their beliefs by assigning an appropriate weight to the arrival of the technical signal given their observation. Because the long-term traders understand only the average behaviors of prices over their observational partitions, they always become more optimistic given positive returns as long as the technical signal arrives with a small probability. Consequently, the long-term traders excessively extrapolate past returns in case the technical signal indeed arrives without their knowledge. Consequently, a feedback loop of price changes emerges because the long-term traders’ positive feedback trading gets amplified over time by its own effects.

As in the benchmark case, there always exists a fully-revealing CLEE. I focus on the case of a learning bubble equilibrium that has a partially-revealing CLEE. I call the collection of trading times in which a feedback loop of upward price changes arises a ‘bubble period’, and the collection of all other trading times a ‘non-bubble period’. I denote the bubble period to be \( T^* \equiv \{\tau, \tau + 1, \ldots, \tau + b - 1\} \) where \( b \geq 1 \) is the length of the bubble period. I call the price path at time \( t = \tau - 1 \) that triggers the bubble period in the future a ‘trigger path’, which is denoted by \( \lambda_{\tau - 1}^b \) \( \cdot \) I call any price path during the bubble period a ‘bubble path’, and denote \( \Lambda_t^* \) to be the set of all possible bubble paths at time \( t \). I also denote \( \hat{\Lambda}_t = \Lambda_t \setminus \Lambda_t^* \) to be the set of all ‘non-bubble paths’, which is a set of price history void of any bubble path.\(^{28}\) The timing of a learning bubble is summarized in Figure 4.

I give the following proposition by relegating the proof to Appendix E.

**Proposition 3 (Learning Bubble Equilibrium)** When \( \eta \) is small enough, there exists a learning bubble equilibrium under some parameter values such that the following properties are true for the set of bubble paths \( \{\Lambda_t^*\}_{t=\tau}^{\tau+b-1} \) and a trigger path \( \lambda_{\tau-1}^b \in \Lambda_{\tau-1} \) with \( Pr(\lambda_{\tau-1}^b \mid B) > 0 \) and \( \frac{\partial Pr(\lambda_{\tau-1}^b \mid B)}{\partial \eta} > 0 \).

**I. For all non-bubble paths** : 

\( \lambda_t \in \hat{\Lambda}_t \):

enough history partitions \( \mathcal{F}_t \) which contains each price path in separate history partition. Since real numbers are dense, there exists such sequence of real numbers \( -\infty < r_1^t < r_2^t < \ldots < r_N^t < \infty \) which separates all possible realizations of returns for each period, and this constructs a return partition given in Definition 1 for each period.

\(^{27}\)Note that the return at \( \tau - 1 \) needs to be greater than or equal to zero (i.e., \( s_{\tau - 1} \geq 0 \)) due to Definition 6.

\(^{28}\)The relative complement of a set \( A \) in a set \( B \) is defined to be \( B \setminus A = \{ z \in B \mid z \notin A \} \).
II. For all bubble paths $\lambda_t \in \Lambda^*_t$: 

1. **State-independent Return Process:** $p_t > p_{t-1}$ for all $\epsilon_t \in \{--, -, \emptyset, +, ++\}$,

2. **Positive Momentum of Expected Return:** $E_t[p_{t+1}] > p_t$.

The two properties in the case of the non-bubble paths are exactly the same as the benchmark case: the State-dependent Return Process Property (Property-S-I) and the Momentum of Expected Return Property (Property-M-I). On the other hand, the two properties in the case of the bubble paths are different from the benchmark case: the State-independent Return Process Property (Property-S-II) and the Positive Momentum of Expected Return Property (Property-M-II). In summary, there occurs a feedback loop of upward price changes when $\lambda_t \in \Lambda^*_t$. Figure 5 shows simulated examples of a learning bubble equilibrium that follows the four properties.

As in the benchmark case, I first solve for the optimal choices of the traders given the four properties. Then, I show that the equilibrium that emerges out of those optimal choices indeed has these properties. For notational convenience, I denote $\kappa(\lambda_{\tau-1}^b) \equiv Pr(\lambda_{\tau-1} = \lambda_{\tau-1}^b | B)$ to be the probability of the arrival of the trigger path given $\omega = B$. Using Property-S-I and Property-S-II, I solve for the evolution of long-term traders’ beliefs in each period.

**Lemma 3** The evolution of the long-term traders’ beliefs is given by the following linear equation with the initial condition $l_1 = \log(\pi/(1-\pi))$: 

$$ l_t = l_{t-1} + \Delta l_{s,t} $$
Figure 5. Simulated examples of learning bubble equilibrium without reaching the trigger path (left) and with reaching the trigger path (right) (The trigger path for a bubble is three consecutive periods of $\epsilon_t \in \{-\} \text{ at each time } t = 98, 99, 100 \text{ (} \kappa(\lambda^b_{t-1}) = \eta^p\text{)}, Parameter values: $\omega = B, T = 500, \theta = 0.08, M = 0.1, \bar{y} = 0.01, \gamma = 1, \pi = \frac{1}{2}, \eta = \rho = 0.001, b = 20\)
Because $\Phi_{t-1}$ is monotone increasing in $\kappa(\lambda_{\tau-1}^b)$, the evolution of the long-term traders’ beliefs becomes similar to the benchmark case as $\kappa(\lambda_{\tau-1}^b)$ becomes smaller. I confine the analysis to the case where $\kappa(\lambda_{\tau-1}^b)$ is a function of $\eta$ such that $\frac{\partial \kappa}{\partial \eta} > 0$. Then, the evolution of the long-term traders’ beliefs becomes sufficiently close to the benchmark case for small enough $\eta$.\footnote{$\Phi_{t-1}(\lambda_t)$ approaches to zero as $\eta \downarrow 0$.}

For example, there exists such a function $\kappa(\lambda_{\tau-1}^b) = \eta^{\tau-1}$, if a learning bubble is triggered by $\tau - 1$ consecutive upward price movements driven by $\epsilon_t \in \{-\}$ at each time $t \in \{1, 2, \ldots, \tau - 1\}$.

The long-term traders’ aggregate demand at time $t$ is given by $l_t$ and $p_t$. The aggregate demand of the speculators is given by the same strategy as in the benchmark case if $\lambda_t \in \hat{\Lambda}_t$, because of Property-S-I and Property-M-I. However, when $\lambda_t \in \Lambda_t^*$, the aggregate demand of the speculators is given by $x_t^I = M$. That is, the demand becomes a price-inelastic function because the risk-neutral speculators expect the return will be strictly positive due to Property-M-II.\footnote{Note that $\epsilon_t$ is the only source of randomness to the speculators. Therefore, each history path $\lambda_t$ is unique to each realization of $\epsilon_t$ given $\omega$.}

Suppose that the trigger path has been reached at time $t = \tau - 1$, i.e., $\lambda_{\tau-1} = \lambda_{\tau-1}^b$. Then, the following results from Lemma.\footnote{The speculators in an aggregate level take the maximum long position at $\tau - 1$ and unwind it sometime when adverse liquidity shock arrives after $\tau + b - 1$.}

\[ p_t > p_{t-1} \text{ if } \Delta l_{u,t} > \gamma \bar{y} \text{ for all } t \in T^*. \]

Furthermore, the long-term traders’ belief updates are close to the level in the benchmark case if $\Phi_{t-1}(\lambda_t)$ is small. When $\eta$ is small enough, it is possible to find some region of $\theta$ which satisfies the following:

\[ \Delta l_{u,t} > \gamma \bar{y} \text{ if } p_{t-1} > p_{t-2} \text{ for all } t \in T^*. \]

Because the two conditions in Definition are satisfied, the initial upward price movement at time $t = \tau - 1$ creates a feedback loop of upward price changes during the bubble period. In the benchmark case, the long-term traders’ positive feedback trading is frequently corrected by the speculators depending on the realizations of $\epsilon_t$. On the other hand, the long-term trader’ positive feedback trading is uninterrupted during the bubble period, thereby creating self-reinforcing upward price movements. This finding shows why positive feedback trading might or might not get amplified depending on the situation.
6 Extension

6.1 Crashes

In the learning bubble equilibrium in the previous section, the divergence of prices from the fundamental value is always slowly adjusted over time once the bubble period is over. In real life, more acute corrections such as crashes can occur together. It is well known that competitive traders are often unable to burst bubbles on their own because of a lack of coordination. (See, for example, De Long, Shleifer, Summers, and Waldmann (1990a), Shleifer and Vishny (1997) and Abreu and Brunnermeier (2003)) Similarly, the speculators in this model never engage in the correction of mispricing unless all other speculators coordinate to do so. The coordination in attacking mispricing can be initiated by a technical signal, thereby creating a crash following a learning bubble.

I assume that a crash can occur only when a learning bubble already has occurred. I denote $\lambda_{t-1}^c$ to be the trigger path that triggers the crash at the subsequent period $t = \tau'$. I denote the crash period to be $T^{**} = \{\tau', \tau' + 1, \ldots, \tau' + c - 1\}$ where $c \geq 1$ is the length of the crash period. I denote $\lambda_{\tau'-1}^c$ to be the trigger path that triggers the crash in the subsequent trading time $t = \tau'$. I call each path of the price history during the crash period a ‘crash path’ and denote $\Lambda_t$, $\Lambda_t^{**}$ to be the set of all bubble paths and crash paths at time $t$, respectively. I also denote $\hat{\Lambda}_t \equiv \Lambda_t \setminus (\Lambda_t^* \cup \Lambda_t^{**})$ to be the set of all ‘non-bubble paths’, which is a set of price history void of any crash or bubble path. The timing of a learning bubble with a crash is summarized in Figure 6.

The learning bubble equilibrium with a crash is exactly the same as the one described in Section 5.2 except that there is a set of crash paths $\{\Lambda_t^{**}\}_{t=\tau'}$ and a trigger path $\lambda_{\tau'-1}^c \in \Lambda_{\tau'-1}$ with $Pr(\lambda_{\tau'-1}^c | B) > 0$ and $\frac{\partial Pr(\lambda_{\tau'-1}^c | B)}{\partial \eta} > 0$. Furthermore, there is an additional property for the crash paths as follows:

III. For all crash paths $\lambda_t \in \Lambda_t^{**}$:
Figure 7. A simulated example of a learning bubble with a crash (The trigger path for a bubble is three consecutive periods of $\epsilon_t \in \{-\} \text{ at each time } t = 98, 99, 100 \text{ (} \kappa(\lambda^b_{t-1}) = \eta^3 \text{)}, \text{ and the trigger path for a crash given the bubble period is } \epsilon_t \in \{++\} \text{ at time } t = 150 \text{ (} \kappa(\lambda^c_{t-1}) = \eta \text{)}, \text{ Parameter values: } \omega = B, T = 500, \theta = 0.08, M = 0.1, \bar{y} = 0.01, \gamma = 1, \pi = \frac{1}{2}, \eta = \rho = 0.001, b = 20, c = 15) \text{.)}

(1) State-independent Negative Return Process: $p_t < p_{t-1}$,

(2) Negative Momentum of Expected Return: $E[p_{t+1}] < p_t$.

The State-independent Negative Return Process Property (Property-S-III) states that the return is always negative during the crash period and the Negative Momentum of Expected Return Property (Property-M-III) states that the speculators correctly expect strict negative returns during that period. Note that a crash might or might not happen depending on the sample path of the price history, i.e., a crash occurs only when $\lambda^c_{t-1}$ is reached. Figure 7 shows a simulated example of a learning bubble equilibrium with a crash.

Using Property-S-I, Property-S-II and Property-S-III, I obtain the evolution of the long-term traders’ beliefs in each period.

**Lemma 4** The evolution of the long-term traders’ beliefs is given by the following linear equation:

$$l_t = l_{t-1} + \Delta l_{s,t}$$

where (i) the change in belief:

$$\Delta l_{s,t} = \begin{cases} \log \left( \frac{1+\theta(1+\Phi_{t-1}(\lambda_t))}{1-\theta(1+\Phi_{t-1}(\lambda_t))} \right), & \text{if } s_{t-1} = u_{t-1}; \\ \log \left( \frac{1-\theta(1-\Phi_{t-1}(\lambda_t))}{1+\theta(1-\Phi_{t-1}(\lambda_t))} \right), & \text{otherwise}; \end{cases}$$
and (ii) the probability of a bubble and a crash:

\[
\Phi_{t-1}(\lambda_{t}) = \begin{cases} \\
\frac{\kappa(\lambda_{t-1})}{\kappa(\lambda_{t-1})} \left( \frac{1-\theta}{2} \right)^{t-1} (1-\kappa(\lambda_{t-1})) & \text{if } \Lambda^{*}_{t-1} \subset J_{t-1}(\lambda_{t});
\frac{\kappa(\lambda_{t-1})}{\kappa(\lambda_{t-1})} \left( \frac{1-\theta}{2} \right)^{t+1} (1-\kappa(\lambda_{t-1})) & \text{if } \Lambda^{*}_{t-1} \subset J_{t-1}(\lambda_{t});
0, & \text{otherwise};
\end{cases}
\]

for all \( t = 1, 2, \ldots, T \) and \( \kappa(\lambda_{t-1}) \equiv Pr(\lambda_{t-1}|\Lambda^{*}_{t+b-1}) \).

Proof See Appendix F.

Due to Property-M-III, the aggregate demand of the speculators is given by \( x^{I}_{t} = -M \) for all \( t \in T^{**} \). From Lemma 2, I find that

\[ p_{t} < p_{t-1} \text{ if } \Delta l_{d,t} < -\gamma \bar{y} \text{ for all } t \in T^{**}. \]

Furthermore, the belief updates of the long-term traders are close to the level in the benchmark case if \( \Phi_{t-1}(\lambda_{t}) \) is small. Therefore, it is possible to have the following for some parameter values as long as the trigger path arrives with a small probability:

\[ \Delta l_{d,t} < -\gamma \bar{y} \text{ if } p_{t-1} < p_{t-2} \text{ for all } t \in T^{**}. \]

A feedback loop of downward price changes emerge through a similar mechanism of creating a feedback loop of upward price changes. I relegate the rest of the proof to Appendix F, which establishes the existence of parameter regions that satisfy the additional conditions.

6.2 Slow-growing Learning Bubble

In a learning bubble, prices move in one direction due to feedback loops. Therefore, they are acute price changes, which do not allow any downward fluctuations. In this section, I explore a more realistic type of self-feeding price movements where both upward and downward price movements occur during the bubble period. Even though there are some price fluctuations, the price process tends to show a strong upward trend during the bubble period. A slow-growing learning bubble equilibrium is exactly the same as the one in Section 5.2 except for the second property as follows:

**Proposition 4** (Slow-growing Learning Bubble Equilibrium) *When \( \eta \) is small enough, there exists a slow-growing learning bubble equilibrium under some parameter values such that the*
Figure 8. A simulated example of a slow-growing learning bubble (The trigger path for a bubble is three consecutive periods of $\epsilon_t \in \{-\} \text{ at each time } t = 48, 49, 50$ ($\kappa(\lambda^b_{t-1}) = \eta^b$), Parameter values: $\omega = B, T = 500, \theta = 0.08, M = 0.1, \bar{y} = 0.01, \gamma = 1, \pi = \frac{1}{2}, \eta = \rho = 0.001, b = 150$)

following properties are true for the set of bubble paths $\{\Lambda^*_t\}_{t=\tau}^{\tau+b-1}$ and a trigger path $\lambda^b_{\tau-1} \in \Lambda_{\tau-1}$ with $Pr(\lambda^b_{\tau-1}|B) > 0$ and $\frac{\partial Pr(\lambda^b_{\tau-1}|B)}{\partial \eta} > 0$.

- For all paths $\lambda \in \Lambda_t$:

  **Momentum of Expected Return:** for all $\omega \in \{G,B\}$ (i) $E_t[p_{t+1}] \geq p_t$ if $p_t \geq p_{t-1}$, (ii) $E_t[p_{t+1}] < p_t$ if $p_t < p_{t-1}$.

I. For all non-bubble paths $\lambda_t \in \hat{\Lambda}_t$:

  **State-dependent Return Process:** (i) When $\omega = G$, $p_t \geq p_{t-1}$ if $\epsilon_t \in \{--, -, \emptyset\}$ and $p_t < p_{t-1}$ if $\epsilon_t \in \{+, ++\}$, (ii) When $\omega = B$, $p_t \geq p_{t-1}$ if $\epsilon_t \in \{--, -\}$ and $p_t < p_{t-1}$ if $\epsilon_t \in \{\emptyset, +, ++\}$.

II. For all bubble paths $\lambda_t \in \Lambda^*_t$:

  **Quasi State-dependent Return Process:** $p_t \geq p_{t-1}$ if $\epsilon_t \in \{--, -, \emptyset\}$ and $p_t < p_{t-1}$ if $\epsilon_t \in \{+, ++\}$.

The State-dependent Return Process Property (Property-S) is exactly the same as in learning bubbles, and the Quasi State-dependent Return Process (Property-QS) states that the return process in a bad state behaves in the same way as in the good state once the price history gets into the bubble paths. Figure 8 shows a simulated example of a slow-growing learning bubble equilibrium.
Using Property-S and Property-QS, I obtain the evolution of the long-term traders’ beliefs in each period.

**Lemma 5** The evolution of the long-term traders’ beliefs is given by the following linear equation:

\[ l_t = l_{t-1} + \Delta s_t \]

where (i) the change in belief:

\[ \Delta s_t = \begin{cases} \log \left( \frac{1+\theta}{(1-\Phi_{t-1}(\lambda_t))+(1+\theta)\Phi_{t-1}(\lambda_t)} \right), & \text{if } s_{t-1} = u_{t-1}; \\ \log \left( \frac{1}{(1+\theta)(1-\Phi_{t-1}(\lambda_t))+(1-\theta)\Phi_{t-1}(\lambda_t)} \right), & \text{otherwise}; \end{cases} \]

for all \( t = 1, 2, \ldots, T \) and (ii) the probability of a bubble:

\[ \Phi_{t-1}(\lambda_t) = \begin{cases} e^*(k)\kappa(\lambda_{t-1}^t), & \text{for } t \in T^*; \\ 0, & \text{otherwise}; \end{cases} \]

where \( e^*(k) = t_{k-1}C_k\left( \frac{1+\theta}{2} \right)^{t-\tau-k} \) and \( e_t(k) = t_{k-1}C_k\left( \frac{1-\theta}{2} \right)^{t-\tau-k} \) given \( k \) upward price movements during the period \( \{\tau, \tau+1, \ldots, t-1\} \in T^* \).

**Proof** See Appendix G.

The aggregate demand of the speculators is given by the same bang-bang solution as in the benchmark case. Note that the probability of having an upward price movement given the bad state is \( \frac{1-\theta}{2} \) in non-bubble paths but it is \( \frac{1+\theta}{2} \) in bubble paths. From Lemma 2 I find that:

\[ Pr(p_t > p_{t-1}|\lambda_t \in \Lambda^*_t, B) > Pr(p_t > p_{t-1}|\lambda_t \in \hat{\Lambda}_t, B) \text{ if } \Delta l_{u,t} > 0 \text{ for all } t \in T^*. \]

Furthermore, the belief updates of the long-term traders are close to the level in the benchmark case if \( \Phi_{t-1}(\lambda_t) \) is small. Therefore, it is possible to have the following for some parameter values as long as the trigger path arrives with a small probability:

\[ \Delta l_{u,t} > 0 \text{ if } p_{t-1} > p_{t-2} \text{ for all } t \in T^*. \]

Unlike in a learning bubble, feedback loops are defined as a probabilistic shift of the price process that allows more upward price movements during the bubble period. I relegate the rest of the proof to Appendix G which establishes the existence of parameter regions that satisfy the equilibrium properties.
7 Empirical Implications

7.1 Bubble and Crashes

There have been well-known incidents of bubbles and crashes through the history of financial markets (e.g., the Dutch Tulip Mania (1634-1637), the South Sea Bubble (1719-1720), the Great Crash (1929), the Internet Bubble (1992-2000) and the Credit Crunch (2007)). Although the characteristics of bubbles can vary a lot depending on the economic situations during the incidents, there is some evidence that at least some bubble incidents feature (i) informed traders who ride the bubbles, and (ii) uninformed traders who keep increasing their shares over time, and lose a significant amount of wealth after the collapses of the bubbles. For example, Brunnermeier and Nagel (2004) and Temin and Voth (2004) find that informed traders rode bubbles due to predictable investor sentiment during the Internet Bubble and the South Sea Bubble periods, respectively. Because such phenomena can be well explained by the combination of limited arbitrage and behavioral biases such as optimistic beliefs or investor sentiment (e.g., De Long, Shleifer, Summers, and Waldmann (1990a) and Shiller (2000)), the question still remains where the behavioral biases come from and how the behavioral biases might have survived without being adjusted despite the relatively long history of financial markets. Because biased beliefs are a system of subjective beliefs that is inconsistent with the equilibrium, it should be critically assumed that the bias is not corrected even after many repetitions of the same situation to maintain the stability of the equilibrium. For example, Bray (1982) shows that an equilibrium with subjective beliefs driven by misspecified models is unstable if the misspecification can be adjusted over time. Another interesting observation is that the bubbles and crashes have been repeated through the history of financial markets, but they seem to occur at a very low frequency. Therefore, it raises another question why the behavioral biases affect the financial market intermittently.

The result of this paper shows that bubble-like price patterns arise intermittently due to the feedback effects of price changes driven by excessive extrapolation of past returns. Since gradual amplification of optimism is caused by natural responses to cognitive limits, the equilibrium outcome is stable even in the long run unless there is any significant change in cognitive resources. Furthermore, I demonstrate that the bubble-like price patterns should arise at a low frequency in the equilibrium. Proposition 3 shows that the technical signal that

\[ \text{Bray (1982) finds that the temporary equilibrium with subjective beliefs eventually converges to the rational expectations equilibrium with objective beliefs in the long run if the adjustment of misspecification is allowed.} \]

\[ \text{See Greenwood and Nagel (2009).} \]

\[ \text{One of the possible explanations mentioned by Greenwood and Nagel (2009) is related to the consequence of adaptive learning such that young generations who have not directly experienced stock market downturns are more prone to the optimism that fuels bubbles. While it could be true for some big bubbles which occur not more than once in each generation, the story cannot explain a series of smaller bubbles which occur multiple times in one generation.} \]
triggers financial fads needs to arrive at a low frequency. I summarize the finding in the following corollary.

**Corollary 1** The frequency of financial fads $Pr(\lambda_{t-1}^b | B)$ needs to be low in a learning bubble equilibrium.

Bubbles and crashes in real life are much more complicated economic events which involve many other factors such as consumption and investment in private as well as public sectors. As is shown in many historical examples of financial fads or manias, the feedback mechanism which is studied in this paper will contribute to the formation of bubbles along with other feedback mechanisms such as wealth effects, consumption habits, and credit expansions.

### 7.2 Price Behaviors

#### 7.2.1 Momentum and Reversal

There exists a large volume of empirical evidence regarding short-term momentum effects (e.g., Jegadeesh and Titman (1993), Lakonishok, Shleifer, and Vishny (1994) and Jegadeesh and Titman (2001)) and long-term reversal effects (e.g., De Bondt and Thaler (1985), Lehmann (1990), Lee and Swaminathan (2000) and Jegadeesh and Titman (2001)). Theoretical explanations in general show that delayed response to information can create momentum effects. For example, Daniel, Hirshleifer, and Subramanyam (1998) show that momentum occurs due to delayed overreaction driven by self-attribution bias. Hong and Stein (1999) show that underreaction to information occurs due to slow information diffusion across the population of traders. Vayanos and Woolley (2008) show that momentum occurs because migrations among delegated portfolios create delayed response when investors gradually learn about fund managers’ ability. In this paper, I confirm the findings of the previous literature because the momentum in this paper is also mainly driven by positive feedback trading that is caused by the long-term traders’ delayed learning.

Empirical evidence of Nofsinger and Sias (1999) and Hvidkjaer (2006) finds that investors engage in positive feedback trading, which can contribute to momentum effects. Although momentum effects can be potentially explained by positive feedback trading described in De Long, Shleifer, Summers, and Waldmann (1990b), the inconsistency of reversal following momentum

---

35 In a learning equilibrium, we need to have small enough $\eta$. Since $\frac{\partial Pr(\lambda_{t-1}^b | B)}{\partial \eta} > 0$, $Pr(\lambda_{t-1}^b | B)$ becomes smaller as $\eta$ gets smaller.

36 There are two important frictions that create price momentum in this model: First, the uninformed traders’ response to price history is delayed. Second, the informed traders have portfolio constraints which makes them unable to push the price immediately to the expected level.
poses challenges to this attempt (e.g., Chan, Jegadeesh, and Lakonishok (1996)). The difference in this paper from the previous theoretical literature is that there are two types of momentum effects: (i) momentum effects without feedback loops, and (ii) momentum effects with feedback loops. Momentum generally arises due to positive feedback trading regardless of the occurrence of feedback loops, and momentum is on average not followed by reversal because momentum is on average heading in the right direction. On the other hand, momentum which is caused with feedback loops always lead to reversal. Therefore, this finding fills the gap between the existing theoretical explanations of momentum effects based on positive feedback trading and empirical observations, by showing that momentum might or might not accompany reversal afterwards depending on the existence of feedback loops.

7.2.2 Positive Feedback Trading and Excessive Extrapolation of Past Returns

In this paper, the long-term traders who have delayed learning optimally engage in positive feedback trading because prices are on average moving in the right direction. In any partially-revealing equilibrium, Monotone Likelihood Ratio Properties (MLRP) of $\Delta l_{s,t}$ is a necessary condition whether it is a learning bubble equilibrium or not, i.e.,

$$\Delta l_{d,t} < 0 < \Delta l_{u,t} \text{ for all } t \in T.$$  

MLRP states that the price is more likely to go up in the good state, and also the opposite is true in the bad state. Since the long-term traders cannot learn the private information of the speculators immediately due to their coarse observational learning, prices slowly converge to the true fundamental value over time.

In this paper, I also shed light on excessive extrapolation of past returns, which can contribute to various empirical phenomena. For example, Lakonishok, Shleifer, and Vishny (1994) argue that momentum and reversal can be caused by the suboptimal trading behavior of some investors with naive expectation that puts excessive weight on the recent history of prices. Instead, I argue that the seemingly ‘excessive’ extrapolation which creates positive feedback trading may not be naive after all, and is rather a consequence of natural responses to the scarcity of cognitive resources. The long-term traders understand that they can excessively extrapolate past returns when the speculators shift their trading strategy simultaneously. Since

---

37 Depending on the occurrence of crashes, the speed of reversal will vary.
38 It is a direct consequence from Lemma C.1, Lemma E.5, and Lemma E.6.
39 Empirical evidence such as Beaver, Lambert, and Morse (1980) finds that price information indeed has significant forecasting power regarding fundamentals.
40 This is related to Norman (1972) who shows a distance diminishing learning operator eventually converges to true value.
41 See, for example, Benartzi (2001) for further evidence of excessive extrapolation of past returns.
such event occurs with a low frequency, the long-term traders also recognize that the probability of excessive extrapolation is small. Therefore, they optimally engage in extrapolation of past returns, which can become excessive with a small probability. Therefore, the tendency of excessive extrapolation of past returns will survive even in the long run without being adjusted unless there is a significant improvement in cognitive resources.

7.2.3 Excess Price Volatility and Price Jumps

Another empirical implication of this paper is that excess price volatility can be generated endogenously by feedback loops without any exogenous shocks. It has been well known that price volatility is too high to be justified by fundamental volatility (e.g., Shiller (1981) and LeRoy and Porter (1981)). Shiller (1990) suggests that feedback trading can be responsible for excess volatility. Frankel and Froot (1990) suggest that trading volumes based on technical analysis can be responsible for large fluctuations in exchange rates, which are not justified by fundamentals. In line with the argument of Shiller (1990) and Frankel and Froot (1990), this paper finds that excess volatility can arise without any fundamental reasons or external shocks. This is because feedback loops of price changes are triggered by internally emerging signals in price history. There is well-documented empirical evidence on discontinuous jumps in the asset price (e.g., Eraker (2001)). It has been widely used in many asset pricing models as the jump diffusion process (e.g., Merton (1976) and Bates (1996)). While price jumps are often interpreted as abnormal shocks due to exogenous arrivals of new information, this paper suggests an additional mechanism for generating price jumps internally.

7.3 Technical Analysis

Academics have been skeptical about technical analysis. Nevertheless, technical analysis has been widely used by practitioners for a long time. Recent academic research such as Brock, Lakonishok, and LeBaron (1992) and Lo, Mamaysky, and Wang (2000) finds empirical evidence of significant forecasting power for technical analysis. The result of this paper implies that technical analysis can possess significant power for forecasting future prices because of its self-fulfilling nature. A series of specific amount of returns is defined as technical patterns that signal buying or selling to the informed traders. In real life, however, there are numerous types of technical signals such as chart patterns (e.g., head-and-shoulder) or time-series patterns (e.g., moving average of past prices). My findings imply that technical signals deliver non-fundamental information regarding the timing of the shift of other traders’ strategies rather

---

42 See, for example, Merton (1976).
43 For example, Zhu and Zhou (2009) analyze the usefulness of moving average trading rule in portfolio allocation problems.
Practitioners who perform technical analysis often use terms such as ‘primary trend’ and ‘secondary trend’. Primary trend means price movements in a long-term period (e.g., a year or more), and secondary trend means price movements in the opposite direction of the primary trend in a short-term period. Some secondary trends are popular enough to have a unique name such as ‘bear market rally’, which often means a short-term rebound of prices after a large drop. A bear market rally does not last too long before it is reverts back to long-term downward trends. The result of the previous section gives an intriguing prediction regarding the span of rallies given different kinds of technical indicators. If a rally is triggered by rising prices, it can last longer. On the other hand, if a rally is triggered by falling prices, it can not sustain itself very long. Bull market rallies are sustained more easily than bear market rallies if those rallies are uninformative price movements simply driven by the feedback loop of price changes during learning bubbles. That is, technical patterns hidden in the past price history with an upward trend can support longer rallies than technical patterns hidden in the past price history with a downward trend. I give the following corollary by relegating the proof to Appendix II:

**Corollary 2** *A financial fad triggered by an upward trend is more likely to be sustainable than the one triggered by a downward trend, i.e.,

\[ \kappa(\lambda^d_{\tau-1}) > \kappa(\lambda^u_{\tau-1}) \]

for any trigger paths \(\lambda^d_{\tau-1}\) and \(\lambda^u_{\tau-1}\) such that \(\lambda^u_{\tau-1}\) is identical to \(\lambda^d_{\tau-1}\) except that it has more upward movements than \(\lambda^d_{\tau-1}\) does.*

8 Conclusion

In this paper, I develop a model of financial fads in which traders’ observational learning is constrained by discrete partitions. The traders are boundedly rational in the sense that they are fully aware of their limitations, and the learning is rational under such limits. In particular, the traders only understand the average behavior of prices over the discrete partitions of past price history. Nevertheless, the result of observational learning is correct on average with any given partition.

I call a feedback loop of price changes that is driven by excessive extrapolation of past returns a ‘learning bubble’, which is a manifestation of self-reinforcing financial fads. Learning bubbles cannot occur if both the informed and uninformed traders share the same observational partitions. Suppose that the informed traders’ observational partitions are finer than
those of the uninformed traders. Then, the informed traders can simultaneously shift their trading strategies depending on the arrival of certain price paths while the uninformed traders only recognize the possibility of such events. Unable to know the exact shift in the informed traders’ strategies, the uninformed traders excessively extrapolate past returns. Consequently, a learning bubble emerge because the uninformed traders’ positive feedback trading gets amplified over time by its own effects. I further find that crashes can arise through the same mechanism that creates learning bubbles.

In this paper, prices are on average moving in the right direction. Therefore, the uninformed traders who have delayed learning optimally engage in positive feedback trading. Momentum arises as a result of positive feedback trading, while reversal does not necessarily follow momentum. I find that momentum is always followed by reversal when momentum is driven by feedback loops of price changes. I also argue that seemingly ‘excessive’ extrapolation may not be naive after all, and is rather a consequence of natural responses to scarce cognitive resources. Therefore, the equilibrium outcome driven by excessive extrapolation is stable even in the long run. I also find that technical signals convey information about non-fundamentals such as the timing of rallies rather than information about fundamentals. The technical signals can possess significant power in forecasting future prices because of their self-fulfilling nature.

Many financial phenomena can arise due to traders’ heuristic thinking. If heuristic simplifications are natural responses to limited cognitive resources, the traders will choose optimal heuristics given the equilibrium price process. If so, the equilibrium price process will also be altered by the optimally chosen heuristics. The endogenous relation between the equilibrium price process and the optimal choice of heuristics under cognitive constraints is at the heart of this paper. The optimally chosen heuristics increase individual-level efficiency of trading at the cost of occasional individual-level mistakes. This paper shows that such individual-level mistakes caused by the heuristics generate the amplification effects of mispricing in the aggregate level. This is because the equilibrium prices suffer aggregate-level instability with some probabilities that are small enough to be overlooked by each individual trader. My findings shed light on intermittently-arising financial instability caused by naive trend-chasing behaviors in various types of markets. For example, future work can explore trend-chasing behaviors in bond or currency markets by incorporating the underlying economics factors of each market within this bounded rationality framework. My findings can give even broader implications beyond financial markets regarding the relationship between heuristic judgements and system instability caused by trend-chasing behaviors of economic agents.
Appendices

Appendix A

First, I claim that the posterior belief of the long-term traders conditional on $\lambda_t$ is equivalent to

$$
\mu^U_t(\omega|\lambda_t) = \frac{Pr(J^U_t(\lambda_t)|\omega)Pr(\omega)}{\sum_{\omega'\in\Omega} Pr(J^U_t(\lambda_t)|\omega')Pr(\omega')} \equiv Pr(\omega|J^U_t(\lambda_t)).
$$

From Definition 4, the consistent observational learning of the long-term traders with the history partition $\lambda_t$ is equivalent to

$$
\mu^U_t(\omega|\lambda_t) = \sum_{\lambda'_t\in J^U_t(\lambda_t)} \frac{Pr(\lambda'_t)Pr(\omega|\lambda'_t)}{\sum_{\omega'\in\Omega} Pr(\lambda'_t|\omega')Pr(\omega')}.
$$

Since $\lambda_t$’s are disjoint events in $\Lambda_t$, it is immediate that $Pr(J^U_t(\lambda_t)|\omega) = \sum_{\lambda'_t\in J^U_t(\lambda_t)} Pr(\lambda'_t|\omega)$. Thus,

$$
\mu^U_t(\omega|\lambda_t) = \sum_{\lambda'_t\in J^U_t(\lambda_t)} \frac{Pr(\lambda'_t)Pr(\omega|\lambda'_t)}{\sum_{\omega'\in\Omega} Pr(\lambda'_t|\omega')Pr(\omega')}.
$$

$$
= \sum_{\lambda'_t\in J^U_t(\lambda_t)} \frac{Pr(\lambda'_t)Pr(\omega)}{\sum_{\omega'\in\Omega} Pr(J^U_t(\lambda_t)|\omega')Pr(\omega')}.
$$

$$
= \sum_{\omega'\in\Omega} Pr(J^U_t(\lambda_t)|\omega')Pr(\omega'),
$$

$$
= Pr(\omega|J^U_t(\lambda_t)).
$$

Therefore, the claim is proved. Using the claim and Bayes’ theorem, it is easily shown that

$$
\mu^U_t(\omega|\lambda_t) = \frac{Pr(J^U_t(\lambda_t)|\omega)\pi}{Pr(J^U_t(\lambda_t)|G)\pi + Pr(J^U_t(\lambda_t)|B)(1-\pi)}.
$$

Appendix B

Since the long-term traders solve a static optimization problem in each period, the optimization problem at time $t$ is given by

$$
\max_{x^U_t} E_t^U\left[ -\exp\left(-\gamma W^U_{T+1}\right) \right]
$$

subject to

$$
W^U_{T+1} = W_0 + (V-p_t)x^U_t.
$$

The first order condition yields

$$
-\mu^U_t(G|\lambda_t)(1-p_t)\exp\left(-\gamma(W_0 + (1-p_t)x^U_t)\right) + (1-\mu^U_t(G|\lambda_t))p_t\exp\left(-\gamma(W_0 - p_t x^U_t)\right) = 0.
$$

\[44\] Note that $Pr(\lambda_t) = \sum_{\omega'\in\Omega} Pr(\lambda'_t|\omega')Pr(\omega').$
The second order condition is satisfied as long as $0 < \mu^U(G|\lambda_t) < 1$. Since the long-term traders are identical and present in a unit mass, solving for $x^U_t$ gives the aggregate demand of the long-term traders at time $t$ given $p_t$ as follows:

$$x^U_t = \frac{1}{\gamma} \left[ l_t + \log \left( \frac{1 - p_t}{p_t} \right) \right].$$

The optimal portfolio choice of the speculators could be obtained directly from the fact that they are short-lived, competitive and risk-neutral. Since the speculators are identical and present in a unit mass, solving for $x_I$ gives the aggregate demand of the long-term traders at time $t$ given $p_t$ as follows:

$$x^I_t = \begin{cases} M, & \text{if } E_I[p_{t+1}] > p_t; \\ [-M, M], & \text{if } E_I[p_{t+1}] = p_t; \\ -M, & \text{if } E_I[p_{t+1}] < p_t. \end{cases}$$

Given the demand of both the speculators and the long-term traders, the market clearing condition is given by

$$x^I_t + x^U_t = y_t.$$ 

Therefore, the equilibrium price at time $t$ given the speculators’ portfolio $x^I_t$ is represented as

$$p_t = \frac{1}{1 + \exp \left( -l_t + \gamma(y_t - x^I_t) \right)}.$$ 

In particular, the demand of the speculators at time $t = 0$ and $t = T$ is zero; therefore, the price is given by

$$p_0 = \frac{1}{1 + \exp \left( -l_0 \right)} = \pi;$$

$$p_T = \frac{1}{1 + \exp \left( -l_T + \gamma y_T \right)}.$$ 

Note that the price is a monotone increasing function in $L_t \equiv l_t - \gamma(y_t - x^I_t)$ such that $F(L_t) = \frac{1}{1 + \exp(-L_t)}$ and $F'(\cdot) > 0$. $L_t$ reflects the degree of the long-term traders’ beliefs, the change in the speculators’ aggregate demand and the net supply of the risky asset. Since $L_t$ is an increasing function in $l_t$, one can also find that the price infinitely approaches to one as $l_t$ approaches to infinity and approaches to zero as $l_t$ approaches to negative infinity. That is, the price converges to either one or zero as the long-term traders’ beliefs converge to either $\omega = G$ or $\omega = B$. The price function in $L_t$ is illustrated in Figure B.1.

The return at time $t$ has the same sign as $L_t - L_{t-1} = \Delta l_{s,t} - \dot{\gamma} [e_t - (x^I_t - x^I_{t-1})]$, i.e.,

$$\text{sign}(p_t - p_{t-1}) = \text{sign} \left( \Delta l_{s,t} - \dot{\gamma} [e_t - (x^I_t - x^I_{t-1})] \right).$$
Therefore, the return in the current period is likely to be higher as the evolution of the long-term traders’ beliefs becomes more positive and the innovation to the supply of shares becomes more negative. If the speculators have taken the maximum long position in the previous period, the return is likely to be lower in the current period if other things are equal because the speculators from the previous period have to unwind their positions. That is, the net demand of the speculators would be at best the same or smaller if they have already taken the maximum long position in the previous period.

Appendix C

Proof of the Existence of a Fully-Revealing Equilibrium:
Suppose that there exists a fully-revealing equilibrium. Then, the change in the log-likelihood ratio at time \( t = 2 \) is given by \( \Delta l_{t,2} = \infty, \Delta l_{d,2} = -\infty \). Therefore, \( E_t[p_2] = 1 \) if \( \omega = G \), and \( E_t[p_2] = 0 \) otherwise. By (7), the speculators’ optimal portfolio is therefore \( x_t = M \) if \( \omega = G \), \( x_t = -M \) otherwise. Since the long-term traders at time \( t = 1 \) do not perform any observation inference due to their delayed learning, we have \( l_1 = l_0 \). By Lemma 2, it could be easily shown that \( p_1 \geq p_0 \) for all \( \epsilon_t \) if and only if \( \bar{y} < M \) and \( p_1 < p_0 \) for all \( \epsilon_t \) if and only if \( -\bar{y} > -M \).

Proof of Proposition 1
I will solve for the optimal choices of the traders assuming both Property-S and Property-M, and show that the equilibrium obtained out of those optimal choices indeed has both properties. First, I derive the evolution of the long-term traders’ beliefs under Property-S. From Lemma 1 the log-likelihood ratio of the long-term traders at time \( t \) is given by the following linear equation with the initial condition \( l_1 = \log(\pi/(1 - \pi)) \):

\[
l_t = l_{t-1} + \Delta l_{x,t}, \tag{C.1}\]

\[\text{Figure B.1. Price function in } L_t\]
where

\[
\Delta l_{s,t} = \begin{cases} 
\log \left( \frac{1+\theta}{1-\theta} \right), & \text{if } s_{t-1} = u_{t-1}; \\
- \log \left( \frac{1+\theta}{1-\theta} \right), & \text{otherwise}; 
\end{cases}
\]

for all \( t = 2, 3, \ldots, T \). Since the sizes of the change in the log-likelihood ratio in the case of the upward and downward price movements are symmetric except for their directions for all \( t = 2, 3, \ldots, T \), I denote \( \Delta l = \Delta l_{u,t} = -\Delta l_{d,t} \) for notational convenience. Given the log-likelihood ratio \( l_t \), the long-term traders at time \( t \) demand the aggregate quantity defined in \([6]\). On the other hand, the aggregate demand of the speculators is given by a ‘bang-bang’ solution due to \([7]\) and Property-M for all \( t = 1, 2, \ldots, T - 1 \):

\[
x^t_l = \begin{cases} 
M, & \text{if } p_t \geq p_{t-1}; \\
-M, & \text{otherwise}. 
\end{cases}
\]

Note that the speculators’ strategy is invariant to the past price history \( \lambda_{t-1} \), and only depends on the sign of the current return \( x_t \) (equivalently, the sign of the current price change \( p_t - p_{t-1} \)). Given the aggregate demand of the speculators as well as the long-term traders, the Walrasian auctioneer sets the price which clears the market according to the clearing condition given by \( x^t_l + x^t_u = y_t \). Given the past price history \( \lambda_t \), there exists a unique market clearing price in each state of the world \( \omega \) and each realization of the new innovation to the supply of shares \( \epsilon_t \).\(^{45}\)

<table>
<thead>
<tr>
<th>( \omega = G )</th>
<th>( \epsilon_t = - )</th>
<th>( \epsilon_t = \emptyset )</th>
<th>( \epsilon_t = + )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{1+\exp(-l_t+\gamma(y_{t-1}-y-M))} )</td>
<td>( \frac{1}{1+\exp(-l_t+\gamma(y_{t-1}-y-M))} )</td>
<td>( \frac{1}{1+\exp(-l_t+\gamma(y_{t-1}+y+M))} )</td>
<td>( \frac{1}{1+\exp(-l_t+\gamma(y_{t-1}+y+M))} )</td>
</tr>
<tr>
<td>( \frac{1}{1+\exp(-l_t+\gamma(y_{t-1}-y-M))} )</td>
<td>( \frac{1}{1+\exp(-l_t+\gamma(y_{t-1}+y-M))} )</td>
<td>( \frac{1}{1+\exp(-l_t+\gamma(y_{t-1}+y+M))} )</td>
<td>( \frac{1}{1+\exp(-l_t+\gamma(y_{t-1}+y+M))} )</td>
</tr>
</tbody>
</table>

Table C.1. Feasible Equilibrium Prices given \( \omega \) and \( \epsilon_t \) for \( t = 1, 2, \ldots, T - 1 \)

Since \( \Delta l_{s,t} \) is only dependent on the previous return \( s_{t-1} \in \{u_{t-1}, d_{t-1}\} \), the market clearing price at time \( t \) is uniquely determined by \( \omega, \epsilon_t \) and \( s_{t-1} \). Given such equilibrium price movements, the speculators, who are also able to learn from the price movement either up or down, can infer whether \( \epsilon_t \in \{-, \emptyset\} \) or \( \epsilon_t \in \{+\} \) if \( \omega = G \). On the other hand, they infer \( \epsilon_t \in \{-\} \) or \( \epsilon_t \in \{\emptyset, +\} \) if \( \omega = B \).

To verify Property-S, it is sufficient to show that the return at the current period indeed follows such property according to the equilibrium prices found in Table D.1.

\(^{45}\)There could potentially exist other market clearing prices given the speculators’ bang-bang trading strategy. However, those prices are in the out-of-equilibrium path. Suppose that the market is cleared at one of those prices in the out-of-equilibrium path. Because the price in the out-of-equilibrium path realizes the current return in the opposite sign to the one in the equilibrium path, Property-S is not true any more. Since the equilibrium price properties are common knowledge, the long-term traders’ beliefs are not updated by \([C.1]\). Therefore, the price in the out-of-equilibrium path would not be sustainable because of the changes in the long-term traders’ portfolio choices.
Lemma C.1 Property-S is true if and only if

$$0 < \Delta l < 2\gamma M.$$  \hfill (C.2)

Proof Since \( p_l \) is uniquely determined by \( \omega, \epsilon_t \) and \( \Delta l_{s,t} \), I denote the price at each state \( \omega, \epsilon_t \) at time \( t \) to be \( p_l^{\omega,\epsilon_t}(\lambda_t) \). Since \( l_0 = l_1 \), it is easily shown that Property-S is trivially true at time \( t = 1 \).

Therefore, Property-S is true if and only if the followings are true for all \( t = 2, 3, \ldots, T \):

(i) \( p_l^{G,\epsilon_t}(s_{t-1}) \geq p_{l-1} \) for all \( \epsilon_t \in \{\emptyset, +\} \) for all \( s_{t-1} \in S_{t-1} \);

(ii) \( p_l^{G,\epsilon_t}(s_{t-1}) < p_{l-1} \) for all \( \epsilon_t \in \{-\} \) for all \( s_{t-1} \in S_{t-1} \);

(iii) \( p_l^{B,\epsilon_t}(s_{t-1}) \geq p_{l-1} \) for all \( \epsilon_t \in \{+\} \) for all \( s_{t-1} \in S_{t-1} \);

(iv) \( p_l^{B,\epsilon_t}(s_{t-1}) < p_{l-1} \) for all \( \epsilon_t \in \{\emptyset, +\} \) for all \( s_{t-1} \in S_{t-1} \);

Using Lemma 2 it could be easily shown that the followings are equivalent to each of the above conditions:

(i) \( \Delta l_{s,t} - \gamma[\epsilon_t - (M - x_{t-1}^s(s_{t-1}])] \geq 0 \) for all \( \epsilon_t \in \{\emptyset, +\}, s_{t-1} \in S_{t-1} \);

(ii) \( \Delta l_{s,t} - \gamma[\epsilon_t - (M - x_{t-1}^s(s_{t-1}))] < 0 \) for all \( \epsilon_t \in \{-\}, s_{t-1} \in S_{t-1} \);

(iii) \( \Delta l_{s,t} - \gamma[\epsilon_t - (M - x_{t-1}^s(s_{t-1}))] \geq 0 \) for all \( \epsilon_t \in \{+\}, s_{t-1} \in S_{t-1} \);

(iv) \( \Delta l_{s,t} - \gamma[\epsilon_t - (M - x_{t-1}^s(s_{t-1}))] < 0 \) for all \( \epsilon_t \in \{-, \emptyset\}, s_{t-1} \in S_{t-1} \);

Equivalently,

(i) \( 0 \leq \Delta l \leq 2\gamma M \);

(ii) \( -\gamma \bar{y} < \Delta l < \gamma(\bar{y} + 2M) \);

(iii) \( -\gamma \bar{y} \leq \Delta l \leq \gamma(\bar{y} + 2M) \);

(iv) \( 0 < \Delta l < 2\gamma M \);

Therefore, the necessary and sufficient condition for satisfying all (i),(ii),(iii),(iv) is

$$0 < \Delta l < 2\gamma M,$$

for all \( t = 2, 3, \ldots, T \). ⊣

To verify Property-M is true, it is sufficient to show that the expected return in the subsequent period indeed follows such property according to the equilibrium prices found in Table D.1.

Lemma C.2 There exist \( M \) such that Property-M is true for all \( M < \tilde{M} \) if

$$\gamma \theta M + \delta < \Delta l < 2\gamma M.$$  \hfill (C.3)

for some constant \( \delta \in (0, \gamma(1 - \theta)M) \).

Proof It is easily shown that Property-M is trivially true at time \( t = T \) if is true for all \( t = 1, 2, 3, \ldots, T - 1 \). Therefore, Property-M is always true if and only if the followings are true for all \( t = 1, 2, 3, \ldots, T - 1 \):

(i) \( p_l^{G,\epsilon_t}(s_{t-1}) \leq \frac{1-\theta}{2} p_{l+1}^G(u_t) + \theta p_{l+1}^G(u_t) + \frac{1-\theta}{2} p_{l+1}^G(u_t) \) for all \( \epsilon_t \in \{\emptyset, +\}, s_{t-1} \in S_{t-1} \);

(ii) \( p_l^{G,\epsilon_t}(s_{t-1}) > \frac{1-\theta}{2} p_{l+1}^G(d_t) + \theta p_{l+1}^G(d_t) + \frac{1-\theta}{2} p_{l+1}^G(d_t) \) for all \( \epsilon_t \in \{-\}, s_{t-1} \in S_{t-1} \);

(iii) \( p_l^{B,\epsilon_t}(s_{t-1}) \leq \frac{1-\theta}{2} p_{l+1}^B(u_t) + \theta p_{l+1}^B(u_t) + \frac{1-\theta}{2} p_{l+1}^B(u_t) \) for all \( \epsilon_t \in \{+\}, s_{t-1} \in S_{t-1} \);
Therefore, there exists \( l(2 + 3\) for small enough \( \Delta \) satisfies the trading horizon is finite and the number of realizations of random variable \( L \) set which includes all possible realizations of \( L \) for small enough \( \Delta \) \( \).

By taking the maximum of \( M \)’s for each possible \( L \), there exists \( M' \) for all \( L \) which satisfies

\[
|\mathfrak{R}_2(L_{t+1})| \leq M' \frac{\Delta L_{t+1}^2}{2},
\]

for small enough \( \Delta L_{t+1} \). Furthermore, there are only finite number of possible \( L_{t+1} \)’s because the trading horizon is finite and the number of realizations of random variable \( y_t \) is also finite. Let \( L_T \) be a set which includes all possible realizations of \( L_t \) given the finite trading horizon \( T \) in equilibrium, i.e.,

\[
L_T = \{ l_t - \gamma(y_t - x_t^l) \mid l_t \in \{-\tau \Delta l, 0, \tau \Delta l\}_T, \forall y_t \in \{ -\tau \tilde{y}, 0, \tau \tilde{y}\}_T, \forall x_t \in \{ -M, M\} \}.
\]

By taking the maximum of \( M' \)’s for each possible \( L_{t+1} \in L_T \), there exists \( M' \) for all \( L_{t+1} \in L_T \) which satisfies

\[
|\mathfrak{R}_2(L_{t+1})| \leq M' \frac{\Delta L_{t+1}^2}{2},
\]

for small enough \( \Delta L_{t+1} \). From Lemma \ref{lemma:C.1} and the assumption such that \( \tilde{y} < M \), we have \( \Delta L_{t+1} \leq (2 + 3\gamma)M \) if Property-S is true. Since \( F'(L_{t+1}) \) is strictly positive and bounded for all \( L_{t+1} \in L_T \), it could be shown that \ref{C.4} and \ref{C.5} are equivalent to \( \Delta l \geq \gamma \theta M + O(M^2) \) if Property-S is true\footnote{\( f(M) = O(M^2) \) as \( M \to 0 \) if and only if there exist positive numbers \( \xi \) and \( m \) such that \( f(M) \leq m|M|^2 \) for all \( |M| < \xi \).}.

Therefore, there exists \( M \) given \( \delta > 0 \) which satisfies Property-M for all \( M < M' \) if

\[
\gamma \theta M + \delta < \Delta l < 2\gamma M.
\]

Finally, the following lemma shows that there exists some parameter regions which satisfy both conditions \ref{C.2} and \ref{C.3}, and it finishes the proof of the existence of partially-revealing equilibrium:
Lemma C.3 There exist some positive numbers $0 < \theta_L < \theta_U < 1$ such that a partially-revealing equilibrium exists for all $\theta \in (\theta_L, \theta_U)$ and for any given $M < \bar{M}$.

Proof Pick a $M^* \in (0, \bar{M})$. Let $\Delta l(\theta) \equiv \log \left( \frac{1 + \theta}{1 - \theta} \right)$. Then, $\Delta l$ is continuous and monotone increasing in $\theta \in (0, 1)$, and $\Delta l(0) = 0, \Delta l(1) = \infty$. Also, note that $\Delta l' > 0$ and $\Delta l'' > 0$. By the intermediate value theorem, there exist a unique $0 < \theta_L < \theta_H < 1$ such that $\Delta l(\theta_L) = \gamma \theta M^* + \delta$ and $\Delta l(\theta_H) = 2 \gamma M^*$. Consequently, $\gamma \theta M^* + \delta < \Delta l(\theta) < 2 \gamma M^*$ for all $\theta \in (\theta_L, \theta_H)$. Therefore, we conclude that given any $M^* < \bar{M}$ there always exists a pair $(\theta_L, \theta_H)$ such that $\gamma \theta M^* + \delta < \Delta l(\theta) < 2 \gamma M^*$ for all $\theta \in (\theta_L, \theta_H)$. □

Appendix D

In a CLEE, the optimal demand of the speculators is a function from the current price to the set of portfolio choices given the price history and the true state of the world, i.e.,

$$x^I_t(\cdot|\lambda_t \in \Lambda_t, \omega \in \Omega) : P_t \to [-M, M]. \quad (D.1)$$

Since the speculators maximize their expected utility conditional on their information set, the optimal demand function $(D.1)$ is equivalent to

$$x^I_t(\cdot|J \in J^I_t(\lambda_t), \omega \in \Omega) : P_t \to [-M, M].$$

That is, the set of the optimal aggregate demand of the speculators given history $\lambda_t$ is identical to the optimal demand given all other possible histories $\lambda_t' \in J^I_t(\lambda_t)$ as long as the true state is the same. In the symmetric equilibrium, the equilibrium strategy is therefore unique to the pair of the given observational partition and state. The following theorem summarizes the findings:

Lemma D.4 The aggregate demand of the speculators is uniquely determined given $J^I_t(\lambda_t)$ and $\omega \in \Omega$.

Proof of Proposition 2

Suppose that there exists a learning bubble equilibrium which satisfy (i) and (ii) in Definition 6. Suppose that the realized state of the world is the bad state, and the learning bubble has indeed occurred in the current sample path of prices, i.e. (i) is true. The feedback loop of price changes in a learning bubble drives the price to go up for sure during the period $T^* = \{\tau, \tau + 1, \ldots, \tau + b - 1\}$, and the speculators know it because it is common knowledge in equilibrium. Therefore, the speculators will take the maximum long position according to $(7)$ as long as the learning bubble persists. Since $x^I_t = M$ for all $t \in T^*$, Lemma 2 shows that the return during the bubble period is always positive as long as $\Delta l_{s,t} > \gamma \bar{y}$ for all $t \in T^*$. That is, the long-term traders’ belief updates should be optimistic enough so that the learning bubble may persist.

Let $\lambda_t$ be the price history at time $t$ during the learning bubble, i.e., $t \in T^*$. Then, the optimal strategy of the speculators at time $t$ is unique given the observational partition of the history $\{s_1, s_2, \ldots, s_t\}$ and the state $\omega$ by the result of Lemma D.4. Since the long-term traders in the subsequent period would
also learn from the same observational partition \( J_t(\lambda_{t+1}) = \{s_1, s_2, \ldots, s_t\} = J_t^I(\lambda_t) \), their assessment of the posterior probability of an upward price movement at time \( t \) conditional on the bad state and the observed price history is given by

\[
Pr(u_t|J_t(\lambda_{t+1}), B) = Pr(\Delta l_{u,t} \geq \gamma \epsilon_t) = 1 \geq Pr(u_t|J_t(\lambda_{t+1}), G),
\]

for any feasible equilibrium strategy of the speculators in the good state. This leads to

\[
\Delta l_{u,t+1} = \log \left( \frac{Pr(u_t|J_t(\lambda_{t+1}), G)}{Pr(u_t|J_t(\lambda_{t+1}), B)} \right) \leq 0.
\]

Hence, the condition (i) in Definition 6 is violated. It contradicts the supposition. \( \blacksquare \)

**Appendix E**

**Proof of Lemma 3**

The probability assessment of the long-term traders consistent with Property-S-I and Property-S-II is given by

\[
Pr(u_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1+\theta}{2}, \quad Pr(d_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1-\theta}{2},
\]

and

\[
Pr(u_{t-1}|J_{t-1}(\lambda_t), B) = Pr(u_{t-1}|J_{t-1}(\lambda_t), B)Pr(\hat{\lambda}_{t-1}|J_{t-1}(\lambda_t), B) + Pr(u_{t-1}|\Lambda_{t-1}^*, J_{t-1}(\lambda_t), B)Pr(\Lambda_{t-1}^*|J_{t-1}(\lambda_t), B) = \frac{1-\theta}{2}(1 - \Phi_{t-1}(\lambda_t)) + \Phi_{t-1}(\lambda_t),
\]

\[
Pr(d_{t-1}|J_{t-1}(\lambda_t), B) = \frac{1+\theta}{2}(1 - \Phi_{t-1}(\lambda_t)),
\]

where \( \Phi_{t-1}(\lambda_t) \equiv Pr(\Lambda_{t-1}^*|J_{t-1}(\lambda_t), B) \) is the long-term traders’ assessment of the probability of being in a bubble path at time \( t-1 \) given the history \( \lambda_t \). Using Bayes’ Theorem, it is shown to be

\[
\Phi_{t-1}(\lambda_t) = \frac{Pr(J_{t-1}(\lambda_t)|\Lambda_{t-1}^*, B)Pr(\Lambda_{t-1}^*|B)}{Pr(J_{t-1}(\lambda_t)|\Lambda_{t-1}^*, B)Pr(\Lambda_{t-1}^*|B) + Pr(J_{t-1}(\lambda_t)|\hat{\lambda}_{t-1}, B)Pr(\hat{\lambda}_{t-1}|B) + Pr(J_{t-1}(\lambda_t)|\hat{\lambda}_{t-1}, B)Pr(\hat{\lambda}_{t-1}|B)}.\]

The probability that \( \lambda_t \in \Lambda_{t-1}^* \) given \( \omega = B \) is equal to \( \kappa(\lambda_{\tau-1}) \) since all bubble paths are initiated by \( \lambda_{\tau-1}^b \). That is, we have

\[
\kappa(\lambda_{\tau-1}^b) \equiv Pr(\lambda_{\tau-1} = \lambda_{\tau-1}^b|B) = Pr(\lambda_t \in \Lambda_{t-1}^*|B), \text{ for all } t \in T^*.
\]

Note that \( \Lambda_{t-1}^* \subset J_{t-1}(\lambda_t) \) implies that the price movements after \( \tau - 1 \) have been upward movements consecutively until \( t - 1 \). Since \( Pr(J_{t-1}(\lambda_t)|\Lambda_{t-1}^*, B) \) is either one if \( \Lambda_{t-1}^* \subset J_{t-1}(\lambda_t) \) or zero otherwise,
it leads to\footnote{Let $J_{b,\tau}^{\ell} = (s_1, s_2, \ldots, s_{\tau-1})$ be the partition which includes $\lambda_{b-1}^t$. Note that the return is always positive during the bubble period in any bubble path. Then, $J_{b,\tau}^{\ell} = (s_1, s_2, \ldots, s_{\tau-1}, u_{\tau}, u_{\tau+1}, \ldots, u_{t-1})$ contains any possible bubble paths at time $t$. Therefore, there exists some element $J_{b,\tau}^{U} \in J_{b,\tau}^{U}$ such that $\Lambda_{b-1}^t \in J_{b,\tau}^{U}$.}

$$\Phi_{t-1}(\lambda_t) = \begin{cases} \frac{\kappa(\lambda_{b-1}^t)}{\kappa(\lambda_{b-1}^t) + \left(\frac{1 - \theta}{1 + \theta}\right)^{1 - \lambda_{b-1}^t}} & \text{if } \Lambda_{b-1}^t \subset J_{t-1}(\lambda_t); \\ 0, & \text{otherwise;} \end{cases}$$

where $\kappa(\lambda_{b-1}^t) \equiv P_r(\lambda_{b-1}^t | B)$. From Lemma \ref{lem:bubble}, the evolution of the long-term traders' beliefs is given by

$$l_t = l_{t-1} + \Delta l_{s,t}$$

where

$$\Delta l_{s,t} = \begin{cases} \log \left(\frac{1 - \theta}{1 + \theta} \Phi_{t-1}(\lambda_t) + 2 \Phi_{t-1}(\lambda_t)\right) & \text{if } s_{t-1} = u_{t-1}; \\ \log \left(\frac{1 - \theta}{1 + \theta}\right) & \text{otherwise;} \end{cases}$$

for all $t = 1, 2, \ldots, T$.\footnote{Note that $\Phi_{t-1}(\lambda_t) = 0$ if $s_{t-1} = d_{t-1}$.}

\textbf{Proof of Proposition 3.}\n
I will prove that the four properties of Proposition \ref{prop:properties} are indeed true in the equilibrium given the aggregate demand of the long-term traders and the speculators. Note that $\Delta l_{s,t}$ is not independent of the past price history $\lambda_{t-1}$ any more if the long-term traders suspect there may exist bubbles, i.e., $\Lambda_t^b \not\subset J_{t-1}(\lambda_t)$. Then, $\Phi_{t-1}(\lambda_t)$ gets bigger over time as long as $\lambda_t \in \Lambda_t^b$ because the long-term traders become more sure about the occurrence of a learning bubble. Therefore, the bubble period needs to finish at some point before the probability $\Phi_{t-1}(\lambda_t)$ becomes too sizable. First, I prove that Property-S-I and Property-M-I are true for all $\lambda_t \in \Lambda_t$ with $\Lambda_t^b \not\subset J_{t-1}(\lambda_t)$. Then, I prove that Property-S-II and Property-M-II are also true for all $\lambda_t \in \Lambda_t^b$.

\textbf{Lemma E.5} There exist $\tilde{M}^I$ such that Property-S-I and Property-M-I are true for all $M < \tilde{M}^I$ when $\lambda_t \in \Lambda_t$ and $\Lambda_t^b \not\subset J_{t-1}(\lambda_t)$ if

$$\gamma \theta M + \delta < \Delta l < 2\gamma M,$$

where $\Delta l \equiv \log \left(\frac{1 + \theta}{1 - \theta}\right)$ and $\delta$ is a positive constant such that $\delta \in (0, \gamma(1 - \theta)M)$.\footnote{Note that $\Phi_{t-1}(\lambda_t) = 0$ if $s_{t-1} = d_{t-1}$.}

\textbf{Proof} Because $\Phi_{t-1}(\lambda_t) = 0$ for all $\lambda_t \in \Lambda_t$ with $\Lambda_t^b \not\subset J_{t-1}(\lambda_t)$, we have $\Delta l \equiv \Delta l_{s,t} = \Delta l_{d,t} = \log \left(\frac{1 + \theta}{1 - \theta}\right)$. Therefore, everything could be proven exactly in the same way as in Lemma C.1. \hfill \square
Lemma E.6  There exist \( \bar{M}^{II} \leq \bar{M}^{I} \) such that Property-S-I, Property-S-II, Property-M-I and Property-M-II are true for all \( M < \bar{M}^{II} \) when \( \lambda_t \in \Lambda_t \) and \( \lambda_t^* \in J_{t-1}(\lambda_t) \) if

\[
\gamma \theta M + \delta < \Delta l_{u,t} < 2\gamma M \quad \text{for all } t \in T^*.
\]

(E.2)

Proof  First, Property-S-II is true if Property-S-I is true for all \( \lambda_t \in \hat{\Lambda}_t \) with \( \lambda_t^* \in J_{t-1}(\lambda_t) \). Because the aggregate demand of the speculators is \( x_t^I = M \) during \( t \in T^* \), Lemma 1 implies that the return at time \( t \in T^* \) is always positive as long as Property-S-I is true. Likewise, Property-M-II is true if Property-M-I is true for all \( \lambda_t \in \hat{\Lambda}_t \) with \( \lambda_t^* \in J_{t-1}(\lambda_t) \). It is because the price always goes up when \( \lambda_t \in \lambda_t^* \) as long as Property-M-I is true even when \( \lambda_t^* \in J_{t-1}(\lambda_t) \). Therefore, it is sufficient to show that Property-S-I and Property-M-I are both true for all \( \lambda_t \in \hat{\Lambda}_t \) with \( \lambda_t^* \in J_{t-1}(\lambda_t) \).

In the same way as in Lemma [C.1], it is easily shown that Property-S-I is true if

\[
0 < \Delta l_{u,t} < 2\gamma M.
\]

(E.3)

and

\[
-2\gamma M < \Delta l_{d,t} < 0.
\]

(E.4)

for all \( \lambda_t \in \Lambda_t \) with \( \lambda_t^* \not\in J_{t-1}(\lambda_t) \).

When \( s_{t-1} = d_{t-1} \), we have \( \Delta l_{s,t} = -\Delta l \) since \( \Phi_{t-1}(\lambda_t) = 0 \). Therefore, the condition given in Lemma [E.5] is sufficient to satisfy Property-M-I in case there exists any downward price movement for \( t \in T^* \). Therefore, we only need to show the sufficient condition for the case \( s_{t-1} = u_{t-1} \). Then, Property-M-I is true if (i) \( p_{t+1}^{G,\epsilon_1}(s_{t-1}) \leq \eta p_{t+1}^{G,++}(u_t) + (\frac{1}{2} - \eta)p_{t+1}^{G,--}(u_t) + \theta p_{t+1}^{G,00}(u_t) \) for all \( \epsilon_1 \in \{-, -\} \), \( s_{t-1} \in S_{t-1} \); (ii) \( p_{t+1}^{B,\epsilon_1}(s_{t-1}) \leq \eta p_{t+1}^{B,++}(u_t) + (\frac{1}{2} - \eta)p_{t+1}^{B,--}(u_t) + \theta p_{t+1}^{B,00}(u_t) \) for all \( \epsilon_1 \in \{-, -\} \), \( s_{t-1} \in S_{t-1} \);

Using the same technique as in the proof of Lemma C.1, we can show that (i) and (ii) are equivalent to

\[
\Delta l_{u,t} \geq \gamma \theta M + O(\Delta L_{t+1}^2) \quad \text{for all } t \in T^*.
\]

(E.5)

For given \( \delta \), there exists small enough \( \bar{M}' \) which satisfies

\[
\gamma \theta M + \delta < 2\gamma M \quad \text{for all } M < \bar{M}'.
\]

Define \( \bar{M}^{II} = \min(\bar{M}' \bar{M}) \). Then, we find that all four properties of Proposition 3 are satisfied if (E.1) and the following are true for all \( M < \bar{M}^{II} \):

\[
\gamma \theta M + \delta < \Delta l_{u,t} < 2\gamma M \quad \text{for all } t \in T^*.
\]

Finally, it suffices to show that there exists an equilibrium if there exists parameter values which satisfies the conditions [E.1] and [E.2]. The following lemma reveals that there exists such parameter
regions, and it finishes the proof of existence of learning bubble equilibrium:

**Lemma E.7** There exist small enough \( \eta \) which guarantees the existence of some positive numbers \( 0 < \theta_L < \theta_U < 1 \) such that a learning bubble equilibrium exists for all \( \theta \in (\theta_L, \theta_U) \) and for any given \( M < M^{II} \).

**Proof** By Lemma C.3 there exists \( 0 < \theta_L < \theta_U < 1 \) such that (E.1) is satisfied for all \( \theta \in (\theta_L, \theta_U) \) and for any given \( M < M^{II} \). Now, I claim that there exists \( 0 < \theta'_L < \theta'_U < 1 \) such that (E.2) is satisfied for all \( \theta \in (\theta'_L, \theta'_U) \) and for any given \( M < M^{II} \) if \( \eta \) is small enough. First, I define \( \Theta_t(\eta) \equiv (\theta_L, \theta_U) \) to be the interval given \( \eta \), which satisfies (E.2) for all \( \theta \in \Theta_t(\eta) \) and \( M < M^{II} \).

I claim that \( \theta_{L,t} \to \theta_L \) and \( \theta_{U,t} \to \theta_U \) as \( \eta \to 0 \). Let \( G_{L,t}(\theta; \eta) \equiv \gamma M + \delta - \Delta_{L,t}(\theta; \eta) \) and \( G_{U,t}(\theta; \eta) \equiv \Delta_{U,t}(\theta; \eta) - \exp(2\gamma M) \). Also, let \( \theta_{L,t}(\eta) \equiv \max\{\{ \theta \in (0,1) | G_{L,t}(\theta; \eta) = 0 \}\} \) and \( \theta_{U,t}(\eta) \equiv \max\{\{ \theta \in (0,1) | G_{U,t}(\theta; \eta) = 0 \}\} \). Note that \( G_{L,t} \) and \( G_{U,t} \) are continuous in \( \theta \) and \( \eta \). Furthermore, \( \Delta_{L,t}(\theta; \eta) \to \Delta_l(\theta) \) as \( \eta \downarrow 0 \) and \( \Delta_{L,t}(\theta; \eta) \to -\Delta_l(\theta) \) as \( \eta \uparrow 0 \). Therefore, there exists a sufficiently small \( \eta \) such that \( \theta_{L,t}(\eta) \in (\theta_L - \sigma, \theta_L + \sigma) \) and \( \theta_{U,t}(\eta) \in (\theta_U - \sigma, \theta_U + \sigma) \) for arbitrarily small \( \sigma > 0 \).

Therefore, \( \Theta_t(\eta) \) converges to \( \Theta \equiv (\theta_L, \theta_U) \) when \( \eta \) is sufficiently small. There exists \( \bar{\eta} \) which makes \( (\cap_{t \in T} \Theta_t(\eta)) \cap \Theta \not\in \emptyset \) for all \( \eta < \bar{\eta} \). Therefore, we can find an interval such that (E.2) is satisfied for all \( \theta \in (\cap_{t \in T} \Theta_t(\eta)) \cap \Theta \) when \( \eta \) is small enough.

**Appendix F**

**Proof of Lemma E.8**

The probability assessment of the long-term traders consistent with Property-S-I and Property-S-II is given by

\[
Pr(u_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1 + \theta}{2}, \quad Pr(d_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1 - \theta}{2},
\]

and

\[
Pr(u_{t-1}|J_{t-1}(\lambda_t), B) = Pr(u_{t-1}|\hat{L}_{t-1}, J_{t-1}(\lambda_t), B)Pr(\hat{L}_{t-1}|J_{t-1}(\lambda_t), B) \\
+ Pr(u_{t-1}|\Lambda^{**}_{t-1}, J_{t-1}(\lambda_t), B)Pr(\Lambda^{**}_{t-1}|J_{t-1}(\lambda_t), B) \\
= 1 - \frac{\theta}{2}(1 - \Phi_{t-1}(\lambda_t)),
\]

\[
Pr(d_{t-1}|J_{t-1}(\lambda_t), B) = 1 + \frac{\theta}{2}(1 - \Phi_{t-1}(\lambda_t)) + \Phi_{t-1}(\lambda_t),
\]

where \( \Phi_{t-1}(\lambda_t) \equiv Pr(\Lambda^{**}_{t-1}|J_{t-1}(\lambda_t), B) \) is the long-term traders’ assessment of the probability of being in a bubble path or a crash path at time \( t - 1 \) given the history \( \lambda_t \). Then,

\[
\Phi_{t-1}(\lambda_t) = Pr(\lambda_t \in \Lambda^{**}_t|J_{t-1}(\lambda_t), \lambda_{t+b-1} \in \Lambda^{**}_{t+b-1}, B)Pr(\lambda_{t+b-1} \in \Lambda^{**}_{t+b-1}|J_{t-1}(\lambda_t), B) \\
+ Pr(\lambda_t \in \Lambda^{**}_t|J_{t-1}(\lambda_t), \lambda_{t+b-1} \not\in \Lambda^{**}_{t+b-1}, B)Pr(\lambda_{t+b-1} \not\in \Lambda^{**}_{t+b-1}|J_{t-1}(\lambda_t), B) \\
= Pr(\lambda_t \in \Lambda^{**}_t|J_{t-1}(\lambda_t), \lambda_{t+b-1} \in \Lambda^{**}_{t+b-1}, B)k_{t-1}^{b}(\lambda_{t-1}^{b}),
\]
where

\[ \kappa^*_t(\lambda^b_{t-1}) = \frac{\kappa(\lambda^b_{t-1})}{\kappa(\lambda^b_{t-1}) + (\frac{1+\theta}{2})^{1+\tau^b}(1-\kappa(\lambda^b_{t-1}))}. \]

Using Bayes’ Theorem leads to

\[ \Phi_{t-1}(\lambda_t) = \begin{cases} \frac{\kappa(\lambda^*_t,\lambda^*_t)\kappa^*_t(\lambda^b_{t-1})}{\kappa(\lambda^*_t,\lambda^*_t)+(\frac{1+\theta}{2})^{1+\tau^b}(1-\kappa(\lambda^*_t,\lambda^*_t))}, & \text{if } \Lambda^*_t \subset J_{t-1}(\lambda_t); \\ 0, & \text{otherwise}; \end{cases} \]

where \( \kappa(\lambda^*_t,\lambda^*_t) \equiv Pr(\lambda^*_t,\Lambda^*_t, B) = Pr(\lambda^*_t,\Lambda^*_t, B). \) The evolution of the long-term traders’ beliefs can be derived similarly in Lemma 1.

**Proof of the Existence of Learning Bubble Equilibrium with a Crash:**

It is easily shown that the following condition needs to be satisfied upon all the conditions for a learning bubble equilibrium with a crash:

\[ -2\gamma M < \Delta d,t < -\gamma \theta M - \delta \quad \text{for all } t \in T^*, \]

where \( \delta \) is a positive constant such that \( \delta \in (0, \gamma(1 - \theta)M). \) The existence of some parameter regions which guarantee the existence of the equilibrium can be established in the same way as in Section 5.2.

**Appendix G**

**Proof of Lemma 5**

The probability assessment of the long-term traders consistent with Property-S and Property-QS is given by

\[ Pr(u_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1+\theta}{2}, \quad Pr(d_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1-\theta}{2}, \]

and

\[ Pr(u_{t-1}|J_{t-1}(\lambda_t), B) = Pr(u_{t-1}|\hat{A}_{t-1}, J_{t-1}(\lambda_t), B)Pr(\hat{A}_{t-1}|J_{t-1}(\lambda_t), B) + Pr(u_{t-1}|\Lambda^*_t, J_{t-1}(\lambda_t), B)Pr(\Lambda^*_t|J_{t-1}(\lambda_t), B) \]

\[ = \frac{1-\theta}{2}(1 - \Phi_{t-1}(\lambda_t)) + \frac{1+\theta}{2} \Phi_{t-1}(\lambda_t), \]

\[ Pr(d_{t-1}|J_{t-1}(\lambda_t), B) = \frac{1+\theta}{2}(1 - \Phi_{t-1}(\lambda_t)) + \frac{1-\theta}{2} \Phi_{t-1}(\lambda_t), \]

where

\[ \Phi_{t-1}(\lambda_t) = \begin{cases} \frac{\varepsilon_t(\lambda_t,\lambda^b_{t-1})}{\varepsilon_t(\lambda_t,\lambda^b_{t-1}) + \varepsilon_t(\lambda^*_t,\lambda^*_t)(1-\kappa(\lambda^*_t,\lambda^*_t))}, & \text{for } t \in T^*; \\ 0, & \text{otherwise}; \end{cases} \]
and \( e^*_t(k) = t - \tau C_k (\frac{1-\theta}{2}) (1-\theta) t - \tau - k \) and \( e_t(k) = t - \tau C_k (\frac{1+\theta}{2}) (1+\theta) t - \tau - k \) given \( k \) upward price movements. The evolution of the long-term traders’ beliefs can be derived similarly in Lemma 1.

Proof of Proposition 4:
It is easily shown that the following condition needs to be satisfied upon all the conditions for a slow-growing learning bubble equilibrium:

\[
\gamma\theta M + \delta < \Delta l_u,t < 2\gamma M \quad \text{for all } t \in T^*,
\]

and

\[
-2\gamma M < \Delta l_d,t < -\gamma\theta M - \delta \quad \text{for all } t \in T^*,
\]

where \( \delta \) is a positive constant such that \( \delta \in (0, \gamma(1 - \theta)M) \). The existence of some parameter regions which guarantee the existence of the equilibrium can be established in the same way as in Section 5.2.

Appendix H

Proof of Corollary
Let \( \lambda^d_{\tau-1} = (r_1, r_2, \ldots, r_{\tau-1}) \) be a trigger path which triggers a learning bubble at time \( t = \tau \). Pick any return in the pattern at arbitrary time \( 1 \leq \hat{\tau} \leq \tau - 1 \), and suppose that it is a downward movement, i.e., \( r_{\hat{\tau}} < 0 \). Also, let \( \lambda^u_{\tau-1} \) be another trigger path which is exactly same but \( r_{\hat{\tau}} \) is replaced by \( r^u_{\hat{\tau}} \), which is an upward movement at time \( t = \hat{\tau} \), i.e., \( r^u_{\hat{\tau}} \geq 0 \). By Proposition 3, returns are independent over time on non-bubble paths in a learning bubble equilibrium. Therefore, the probabilities of reaching those triggers have the following relationship:

\[
\kappa(\lambda^d_{\tau-1}) \equiv Pr(\lambda^d_{\tau-1} | B) = Pr(\lambda^u_{\tau-1} | B) \frac{Pr(d_{\tau} | B)}{Pr(u_{\tau} | B)} = \kappa(\lambda^u_{\tau-1}) \frac{1 + \theta}{1 - \theta} > \kappa(\lambda^u_{\tau-1}).
\]

One could easily verify that \( \Delta l_{u,t} \) is decreasing as \( \kappa \) gets larger. Any deviation of \( \Delta l_{u,t} \) from \( \Delta l \) makes the sufficient condition for the existence tighter as is obvious in Lemma E.5 and Lemma E.6. Therefore, a learning bubble equilibrium is hard to exist when \( \kappa \) is larger.

References


Note that \( \frac{\partial \Delta l_{u,t}}{\partial \kappa} < 0 \).


